WEAK-HEAD CONVERSION TESTING FOR MTT

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Abstract. We detail the small set of adaptations required to extend the weak-head conversion testing algorithm used by `Agda` to MTT. We pay attention to the case of preorder-enriched mode theories which are particularly simple and arise frequently in examples.

Contents

1. Introduction

The normalization result from Gratzer [Gra22] provides the theoretical guarantee that MTT can be implemented. Unfortunately, it has been consistently observed that the gap between theory and practice is smaller in theory than in practice. Stassen, Gratzer, and Birkedal [SGB23] details a defunctionalized normalization-by-evaluation algorithm inspired by Gratzer [Gra22]. In this note, we detail an alternative algorithm following Abel, Öhman, and Vezzosi [AÖV17]. Unlike normalization-by-evaluation, this algorithm does not distinguish between syntactic classes of values, normals, neutrals, etc. and does not involve closures. This last point appears to be something of a boon for MTT, where non-trivial 2-cells induce substitutions which are hard to incorporate the NbE framework.[1]

Remark 1.1. We endeavor to closely follow the notations and conventions of Abel, Öhman, and Vezzosi [AÖV17] to minimize friction. ⋄

The rough outline of the algorithm detailed by Abel, Öhman, and Vezzosi [AÖV17] is as follows:

- We define a type-indexed weak-head reduction judgment $\Gamma \vdash t \longrightarrow u : A$. This presumes that A is in weak-head normal form (whnf)
- We also define a conversion testing judgment $\Gamma \vdash t \iff u : A$ which presumes t, u, and A are all in whnf.

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[1]The issue is comparable to those which arise in the context of cubical type theory and cofibrations. Some progress has been made by Hu and Pientka [HP23] on the matter in a less general context.

- Both of the judgments are twinned so as to also apply to types: $\Gamma \vdash A \longrightarrow B$ and $\Gamma \vdash A \Longleftrightarrow B$.

Using these four judgments in combination, we are able to define a conversion-testing algorithm:

$$\frac{\Gamma \vdash A \longrightarrow \bar{A} \qquad \Gamma \vdash t \longrightarrow \bar{t} : \bar{A} \qquad \Gamma \vdash u \longrightarrow \bar{u} : A_0 \qquad \Gamma \vdash \bar{t} \Longleftrightarrow \bar{u} : \bar{A}}{\Gamma \vdash M \overset{\frown}{\Longleftrightarrow} N : A}$$

Other judgments arise in the process of defining these four core ones. For instance, we will require a judgment comparing neutral elements of types like Nat (or $\langle \mu \mid A \rangle$).

2. Alterations for the preorder-enriched case

Fix a preorder-enriched category $\mathcal{M}$. We will discuss the necessary changes to extend Abel, Öhman, and Vezzosi [AÖV17] to handle MTT instantiated with $\mathcal{M}$. First and perhaps most obviously, we must extend the grammar of all the syntactic sorts.

Remark 2.1. The defining characteristic of preorder-enriched mode theories is that there is no need to explicitly mind 2-cells: they are uniquely determined by their boundaries. For this reason, we shall have no need to change the syntax of variables. It also allows us to systematically surpress the syntax of 2-cell substitutions and the functorial action of $-.\{\mu\}$. ⋄

Remark 2.2. In Gratzer [Gra23], we have written $\mathsf{let}_\nu\, \mathsf{mod}_\mu(-) \leftarrow t \,\mathsf{in}\, u$ for eliminator for modal types. In this note, we write $\mathsf{modrec}_{\nu;\mu}\, B\, u\, t$ instead to (1) make the motive B for the eliminator explicit and more closely parallel what Abel, Öhman, and Vezzosi [AÖV17] write for e.g., the natural numbers. ⋄

$$\begin{array}{llcl}
\textit{(Exp)} & t, u, v, A, B & ::= & \ldots \mid \mathsf{modrec}_{\nu;\mu}\, B\, u\, t \\
\textit{(Whnf)} & \bar{t}, \bar{A} & ::= & \ldots \mid \mathsf{mod}_\mu(t) \mid \langle \mu \mid A \rangle \\
\textit{(Ne)} & n, m, N, M & ::= & \cdots \mid \mathsf{modrec}_{\nu;\mu}\, B\, u\, n \\
\textit{(Ctx)} & \Gamma, \Delta & ::= & \epsilon \mid \Gamma.(\mu \mid A) \mid \Gamma.\{\mu\}
\end{array}$$

We present a few of the typing rules that have changed from the case of Martin-Löf type theory:

$$\frac{\Gamma = \Gamma_0, (\mu \mid A), \Gamma_1 \qquad |\Gamma_1| = n \qquad \mathsf{Locks}(\Gamma_1) = \nu \qquad \mu \leq \nu}{\Gamma \vdash \mathsf{i}_n : A[\uparrow^{\Gamma_1}]}$$

$$\frac{\Gamma \vdash \sigma : \Delta \qquad \Delta.\{\mu\} \vdash t : A}{\Gamma \vdash \sigma, t : \Delta.(\mu \mid A)} \qquad\qquad \frac{\mu \leq \nu \qquad \Gamma \vdash \sigma : \Delta}{\Gamma.\{\nu\} \vdash \sigma : \Delta.\{\mu\}}$$

$$\frac{\Gamma.\{\mu\} \vdash A \;\mathsf{type}}{\Gamma \vdash \langle \mu \mid A \rangle \;\mathsf{type}} \qquad\qquad \frac{\Gamma.\{\mu\} \vdash t : A}{\Gamma \vdash \mathsf{mod}_\mu(t) : \langle \mu \mid A \rangle}$$

$$\frac{\begin{array}{c}\Gamma.\{\nu \circ \mu\} \vdash A \;\mathsf{type} \qquad \Gamma.(\nu \mid \langle \mu \mid A \rangle) \vdash B \;\mathsf{type} \\ \Gamma.(\nu \circ \mu \mid A) \vdash u : B[\uparrow, \mathsf{mod}_\mu(\mathsf{i}_0)] \qquad \Gamma.\{\nu\} \vdash t : \langle \mu \mid A \rangle\end{array}}{\Gamma \vdash \mathsf{modrec}_{\nu;\mu}\, B\, u\, t : B[\mathsf{id}, t]}$$

Notice that our typing rules for substitutions add a form of "overloading"; the functorial action of $-.\{\mu\}$ is entirely silent. This is valid only because all 2-cells with the same boundary are equal. In other words, because we have no need to keep track of the 2-cells on variables, we have need for modal annotations on substitutions. We shall see this change in Section 3. As substitutions have not changed, there is no real need to alter the $t[\sigma]$ aside from the addition of the new cases:

$$\begin{aligned}
&\mathsf{mod}_\mu(t)[\sigma] = \mathsf{mod}_\mu(t[\sigma])\\
&\langle \mu \mid A\rangle[\sigma] = \mathsf{mod}_\mu(t[\sigma])\\
&(\mathsf{modrec}_{\nu;\mu}\, B\, u\, t)[\sigma] = \mathsf{modrec}_{\nu;\mu}\, (B[\Uparrow \sigma])\, (u[\Uparrow \sigma])\, (t[\sigma])
\end{aligned}$$

It remains to present the additional rules for weak-head reduction and conversion testing (the rules for all previous connectives are unchanged):

$$\frac{\begin{array}{c}\Gamma.(\nu \mid \langle \mu \mid A\rangle) \vdash B\ \mathsf{type}\\ \Gamma.(\nu\circ\mu \mid A) \vdash u : B[\uparrow, \mathsf{mod}_\mu(\mathsf{i}_0)] \qquad \Gamma.\{\nu\}.\{\mu\} \vdash t : A\end{array}}{\Gamma \vdash \mathsf{modrec}_{\nu;\mu}\, B\, u\, \mathsf{mod}_\mu(t) \longrightarrow u[\mathsf{id}, t] : B[\mathsf{id}, t]}$$

$$\frac{\begin{array}{c}\Gamma.(\nu \mid \langle \mu \mid A\rangle) \vdash B\ \mathsf{type}\\ \Gamma.(\nu\circ\mu \mid A) \vdash u : B[\uparrow, \mathsf{mod}_\mu(\mathsf{i}_0)] \qquad \Gamma.\{\nu\circ\mu\} \vdash t : A \qquad \Gamma.\{\nu\circ\mu\} \vdash t \longrightarrow t' : A\end{array}}{\Gamma \vdash \mathsf{modrec}_{\nu;\mu}\, B\, u\, t \longrightarrow \mathsf{modrec}_{\nu;\mu}\, B\, u\, t' : B[\mathsf{id}, t]}$$

$$\frac{\Gamma.\{\mu\} \vdash A \hat{\Longleftrightarrow} B}{\Gamma \vdash \langle \mu \mid A\rangle \Longleftrightarrow \langle \mu \mid B\rangle} \qquad \frac{\Gamma.\{\mu\} \vdash A \longrightarrow^* \bar{A} \qquad \Gamma.\{\mu\} \vdash t \hat{\Longleftrightarrow} u : \bar{A}}{\Gamma \vdash \mathsf{mod}_\mu(t) \Longleftrightarrow \mathsf{mod}_\mu(u) : \langle \mu \mid A\rangle}$$

$$\frac{\begin{array}{c}\Gamma.(\nu \mid \langle \mu \mid A\rangle) \vdash B_0 \hat{\Longleftrightarrow} B_1\\ \Gamma.(\nu\circ\mu \mid A) \vdash u_0 \hat{\Longleftrightarrow} u_1 : B_0[\uparrow, \mathsf{mod}_\mu(\mathsf{i}_0)] \qquad \Gamma.\{\nu\} \vdash t_0 \hat{\longleftrightarrow} t_1 : \langle \mu \mid A\rangle\end{array}}{\Gamma \vdash \mathsf{modrec}_{\nu;\mu}\, B_0\, u_0\, t_0 \hat{\longleftrightarrow} \mathsf{modrec}_{\nu;\mu}\, B_1\, u_1\, t_1 : B_0[\mathsf{id}, t_0]}$$

We do not believe any of these rules offer particular surprises; they follow the pattern of other connectives.

3. The general case

In this section, we lift the restriction that the mode theory be merely preorder enriched and consider proper 2-categories. This actually causes some complications: we can no longer ignore 2-cells and must track them through our algorithm. We require a few alterations to our syntax in order for this to work. While terms and contexts remain the same (1) we must change variables so as to record the 2-cell used to access them (2) we tweak substitutions to record the functorial action of modalities, and (3) we require an additional operation on terms to “shift” them. We note that we still do not alter substitutions per se to record modalities, but rather explicitly separate out the key substitutions induced 2-cells as they are only required in a few particular places.

$$\begin{array}{llll}
\textit{(Sb)} & \sigma & ::= & \sigma, \mu\\
\textit{(Ne)} & n, m, N, M & ::= & \ldots \mid \mathsf{i}_i^\alpha
\end{array}$$

Remark 3.1. We emphasize that—unlike with MTT qua GAT—we do not actually view $\Gamma.\{\mu\circ\nu\}$ and $\Gamma.\{\mu\}.\{\nu\}$ as identical. We instead will ensure that there is an admissible rule ensuring that a term is well-typed in one iff it is well-typed in the other. However, we will use this more rigid distinction to avoid carrying around some information in substitutions. We likewise do not insist that $\Gamma.\{\mathsf{id}\} = \Gamma$, though this is less pressing. $\diamond$

In order to make what follows more intelligible, we introduce a new syntactic category of "skeletal contexts". These will contain only a small fraction of the information of an actual context and isolate what is necessary for the shifting operation:

$$\textit{(SkCx)} \quad S, T \quad ::= \quad \epsilon \mid S.\{\mu\} \mid S.\bullet$$

In particular, these contexts erase the types and modal annotations, but retain the information of the modal restrictions and the places where variables appear. There is an evident erasure operation $\Gamma \mapsto \|\Gamma\|$ which erases the types from Γ to produce a skeletal context.

The typing rule for a variable must be correspondingly altered to account for this new syntax:

$$\frac{\Gamma \vdash \sigma : \Delta}{\Gamma.\{\mu\} \vdash \sigma, \mu : \Delta.\{\mu\}}$$

$$\frac{\begin{array}{c}\Gamma = \Gamma_0, (\mu \mid A), \Gamma_1 \\ |\Gamma_1| = n \qquad \mathsf{locks}(\Gamma_1) = \nu \qquad \alpha : \mu \longrightarrow \nu \qquad \|\Gamma_0\|; \alpha \vdash A \rightsquigarrow A'\end{array}}{\Gamma \vdash \mathsf{i}^{\alpha}_{n} : A'[\uparrow^{\Gamma_1}]}$$

The first judgment takes the place of the "subtyping" that we used in Section 2.

In the above, we have made use of a new judgment $\|\Gamma\|; \alpha; \|\Delta\| \vdash A \rightsquigarrow B$ which roughly takes a type $\Gamma.\{\mathsf{dom}(\alpha)\}.\Delta \vdash A\,\mathsf{type}$ and uses α to "shift" all the variables in A to obtain $\Gamma.\{\mathsf{cod}(\alpha)\}.\Delta^{\alpha} \vdash B\,\mathsf{type}$. In this last expression we have written Δ^{α} for the pointwise application of this shifting operation to Δ. We now describe the rules for this judgment along with the corresponding operation for terms.

$$\boxed{S; \alpha; T \vdash t \rightsquigarrow u \qquad S; \alpha; T \vdash A \rightsquigarrow B}$$

$$\frac{S; \alpha; T.\{\mu\} \vdash A \rightsquigarrow B}{S; \alpha; T \vdash \langle \mu \mid A \rangle \rightsquigarrow \langle \mu \mid B \rangle} \qquad \frac{S; \alpha; T \vdash A \rightsquigarrow C \qquad S; \alpha; T.\bullet \vdash B \rightsquigarrow D}{S; \alpha; T \vdash \prod_A B \rightsquigarrow \prod_C D}$$

$$\frac{S; \alpha; T.\{\mu\} \vdash t \rightsquigarrow u}{S; \alpha; T \vdash \mathsf{mod}_{\mu}(t) \rightsquigarrow \mathsf{mod}_{\mu}(u)} \qquad \frac{T = T_0.\bullet.T_1 \qquad |T_1| = i}{S; \alpha; T \vdash \mathsf{i}^{\alpha}_{i} \rightsquigarrow \mathsf{i}^{\alpha}_{i}}$$

$$\frac{S = S_0.\bullet.S_1 \qquad |S_1| + |T| = i \qquad \mathsf{locks}(S_1) = \xi_0 \qquad \mathsf{locks}(T) = \xi_1}{S; \alpha; T \vdash \mathsf{i}^{\beta}_{i} \rightsquigarrow \mathsf{i}^{(\xi_0 \star \alpha \star \xi_1) \circ \beta}_{i}}$$

The last rule is the heart of the judgment. It states that if we are attempting to shift a variable by α which is presently been accessed by β, we modify β by post-composing it with α after appropriately whiskering α to have the correct type.

We must also alter the substitution action. In order to accommodate general 2-cells it must also take place with respect to a pair of skeletal contexts. This is less onerous than it might appear at first blush; all instances of the substitution judgment have both contexts readily available. All of judgments for the "substitution judgment" commute the substitution further into a term. The work is, as expected, done at variables.

$$t[\mathsf{id}, \bar{\nu_i} : S \to T] = t$$
$$\mathsf{mod}_\mu(t)[\sigma : S \to T] = \mathsf{mod}_\mu(t[\sigma : S.\{\mu\} \to T.\{\mu\}])$$
$$\mathsf{i}^\alpha_{i+1}[\sigma, t, \overline{\nu_i} : S.\overline{\{\nu_i\}} \to T.A.\overline{\{\nu_i\}}] = \mathsf{i}^\alpha_i[\sigma, \overline{\nu_i} : S.\overline{\{\nu_i\}} \to T.\bullet.\overline{\{\nu_i\}}]$$
$$\mathsf{i}^\alpha_0[\sigma, t, \overline{\nu_i} : S.\overline{\{\nu_i\}} \to T.\bullet.\overline{\{\nu_i\}}] = u \text{ where } S;\alpha \vdash t \rightsquigarrow u$$

The rules from Section 2 now adapt after adding appropriate annotations.

$$\frac{\Gamma.(\nu \mid \langle\mu \mid A\rangle) \vdash B\ \mathsf{type} \qquad \Gamma.(\nu \circ \mu \mid A) \vdash u : B[\uparrow, \mathsf{mod}_\mu(\mathsf{i}_0) : \|\Gamma\|.\bullet \to \|\Gamma\|.\bullet] \qquad \Gamma.\{\nu\}.\{\mu\} \vdash t : A}{\Gamma \vdash \mathsf{modrec}_{\nu;\mu}\, B\, u\, \mathsf{mod}_\mu(t) \longrightarrow u[\mathsf{id}, t : \|\Gamma\| \to \|\Gamma\|.\bullet] : B[\mathsf{id}, t : \|\Gamma\| \to \|\Gamma\|.\bullet]}$$

$$\frac{\Gamma.(\nu \mid \langle\mu \mid A\rangle) \vdash B\ \mathsf{type} \qquad \Gamma.(\nu \circ \mu \mid A) \vdash u : B[\uparrow, \mathsf{mod}_\mu(\mathsf{i}_0) : \|\Gamma\|.\bullet \to \|\Gamma\|.\bullet] \qquad \Gamma.\{\nu \circ \mu\} \vdash t : A \qquad \Gamma.\{\nu \circ \mu\} \vdash t \longrightarrow t' : A}{\Gamma \vdash \mathsf{modrec}_{\nu;\mu}\, B\, u\, t \longrightarrow \mathsf{modrec}_{\nu;\mu}\, B\, u\, t' : B[\mathsf{id}, t : \|\Gamma\| \to \|\Gamma\|.\bullet]}$$

$$\frac{\Gamma.\{\mu\} \vdash A \mathrel{\hat{\Longleftrightarrow}} B}{\Gamma \vdash \langle\mu \mid A\rangle \Longleftrightarrow \langle\mu \mid B\rangle} \qquad \frac{\Gamma.\{\mu\} \vdash A \longrightarrow^* \bar{A} \qquad \Gamma.\{\mu\} \vdash t \mathrel{\hat{\Longleftrightarrow}} u : \bar{A}}{\Gamma \vdash \mathsf{mod}_\mu(t) \Longleftrightarrow \mathsf{mod}_\mu(u) : \langle\mu \mid A\rangle}$$

$$\frac{\Gamma.(\nu \mid \langle\mu \mid A\rangle) \vdash B_0 \mathrel{\hat{\Longleftrightarrow}} B_1 \qquad \Gamma.\{\nu\} \vdash t_0 \mathrel{\hat{\longleftrightarrow}} t_1 : \langle\mu \mid A\rangle \qquad \Gamma.(\nu \circ \mu \mid A) \vdash u_0 \mathrel{\hat{\Longleftrightarrow}} u_1 : B_0[\uparrow, \mathsf{mod}_\mu(\mathsf{i}_0) : \|\Gamma\|.\bullet \to \|\Gamma\|.\bullet]}{\Gamma \vdash \mathsf{modrec}_{\nu;\mu}\, B_0\, u_0\, t_0 \mathrel{\hat{\longleftrightarrow}} \mathsf{modrec}_{\nu;\mu}\, B_1\, u_1\, t_1 : B_0[\mathsf{id}, t_0 : \|\Gamma\| \to \|\Gamma\|.\bullet]}$$

4. Remaining questions

Of course, the discussion above deals only with the core calculus. It remains to consider how all of the features of MTT might interact with higher-level operations like pattern-matching, data types, records, metavariables etc. One feature worth noting: we should forbid the user from pattern-matching on a variable with a non-trivial modal annotation in general.

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