For ♭-annotated types, Segal implies simplicial

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We prove the claim of the title in the context of triangulated type theory.

1 Introduction

Weaver and Licata [WL20] conjectured that (in a related but slightly distinct setting) every cubical space satisfying a formulation of the Segal condition must arise uniquely from a simplicial space (i.e. *is simplicial*). Later Gratzer, Weinberger, and Buchholtz [GWB24] incorrectly suggested $\{(i, j) : \mathbb{I}^2 \mid i \leq j \vee j \leq i\}$ as a counter-example to this claim. The first author of this note pointed out that not only does this "split square" fail to be Segal, but that this failure indicates the germ of an argument that every ♭-annotated Segal type is simplicial. We show this to be the case in the context of triangulated type theory [GWB24].

For the sake of concreteness, we recall the main definitions of our claim.

Axiom 1 (The interval)**.** *There is a small type* $\mathbb{I}$ *equipped with the structure of a bounded distributive lattice* $(\wedge, \vee, 0, 1)$.[1]

Definition 1.1. The 2-simplex Δ^2 is given by $\sum_{i,j:\mathbb{I}} i \geq j$. The (2,1)-horn is the subtype given by the stronger condition $\sum_{i,j:\mathbb{I}} i = 1 \vee j = 0$.

Definition 1.2. A type is Segal if $X^{\Delta^2} \to X^{\Lambda^2_1}$ is an equivalence.

Definition 1.3. A type is simplicial if $\prod_{i,j:\mathbb{I}} \mathsf{isEquiv}(\mathsf{const} : X \to (i \leq j \vee j \leq i \to X))$. The subcategory of simplicial types is (lex) reflective [RSS20] and we denote the induced monad by ◹.

With these definitions to hand, our claim can be stated as follows:

Theorem 1.4. *If* $X :_\flat \mathcal{U}$ *is Segal then* X *is simplicial.*

Remark 1.5. Note that the converse is false with a counterexample given by, e.g. $\mathcal{U}_◹$. Furthermore, the ♭-annotation is necessary: consider $i \leq j \vee j \leq i$ as all propositions are Segal.

This claim relies on only a handful of the axioms of $\mathsf{TT}_◹$. Aside from the axiom postulating $\mathbb{I}$, we require the following three axioms of triangulated type theory:

Axiom 2 (Crisp induction)**.** *For every* μ*, the canonical map* $\mathrm{mod}_\mu(a) = \mathrm{mod}_\mu(b) \to \langle \mu \mid a = b \rangle$ *is an equivalence.*

Axiom 3 (Cubes separate)**.** *A map* $f :_\flat A \to B$ *is an equivalence if and only if the following holds:*

$$(n :_\flat \mathsf{Nat}) \to \mathsf{isEquiv}((f_*)^\dagger : \langle \flat \mid \mathbb{I}^n \to A \rangle \to \langle \flat \mid \mathbb{I}^n \to B \rangle)$$

[1] As $\mathbb{I}$ is in particular a semi-lattice, it is necessarily a homotopy set.

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Axiom 4 (Simplicial stability). *If $A :_\flat \mathcal{U}$ then the following map is an equivalence for all $n :_\flat \mathrm{Nat}$:*

$$(\eta_*)^\dagger : \langle \flat \mid \Delta^n \to A \rangle \to \langle \flat \mid \Delta^n \to \sharp A \rangle$$

In fact, we show in Appendix A that Axiom 4 (previously used only to show that $\mathcal{S}$ and Cat are simplicial) is in fact a consequence of two other axioms postulated by $\mathsf{TT}_\sharp$. We choose to keep it as a primitive in the main body of this note as it is far easier to state and applies more directly.

Our argument proceeds in three steps. First we show that Segal types are *triangulated* (orthogonal to the inclusion $\{(i,j) : \mathbb{I}^2 \mid i \le j \vee j \le i\} \to \mathbb{I}^2$). Next we show that orthogonality to the "split square inclusion" implies orthogonality to higher-dimensional variants. Finally, we show that orthogonality to the subobjects defined by these high-dimensional variants implies simpliciality.

2 Segal types are triangulated

We begin by showing a simpler result: X is orthogonal to $\{(i,j) : \mathbb{I}^2 \mid i \le j \vee j \le i\} \to \mathbb{I}^2$. In the parlance of `agda-synthetic-categories` [ASC], such types are called *triangulated*.

Notation 2.1. We shall extend this terminology slightly and say that a map is *triangulated anodyne* if it left orthogonal to triangulated types. In other words, if it is in the left class generated by $\{(i,j) : \mathbb{I}^2 \mid i \le j \vee j \le i\} \times X \to \mathbb{I}^2 \times X$.

For this argument, we do not need any sort of $\flat$-annotation on X and we deliberately do not restrict ourselves to $\flat$-orthogonality—the stronger condition will be necessary for the next step. For the sake of notation, we write S for the domain of this map.

Lemma 2.2. *If X is Segal, X is triangulated.*

Proof. We shall argue this by showing that the map $\mathrm{S} \longrightarrow \mathbb{I}^2$ is a retract of an inner anodyne map. To this end, we begin with the following section-retraction pair:

$$\begin{aligned} s &: \mathbb{I}^2 \to \Delta^2 \times \mathbb{I} \\ s(a,b) &= (a,a,b) \\ r &: \mathbb{I}^2 \to \Delta^2 \times \mathbb{I} \\ r(i,j,k) &= (i \wedge (k \vee j), k) \end{aligned}$$

We observe that $r \circ s = \mathrm{id}$ immediately. Our next step is to argue that these maps restrict correctly to the following subtypes:

$$\begin{array}{ccccc} \{(i,j) : \mathbb{I}^2 \mid i \le j \vee j \le i\} & \longrightarrow & \{(i,j,k) : \Delta^2 \times \mathbb{I} \mid \Phi(i,j,k)\} & \longrightarrow & \{(i,j) : \mathbb{I}^2 \mid i \le j \vee j \le i\} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{I}^2 & \longrightarrow & \Delta^2 \times \mathbb{I} & \longrightarrow & \mathbb{I}^2 \end{array}$$

Here Φ is the following somewhat complex predicate:

$$\Phi(i,j,k) = i = 1 \vee j = 0 \vee k = 0 \vee k = 1 \vee j \ge k \vee k \ge i \vee i \ge k \ge j$$

It is routine at least to check that the two required factorizations exist. That s lifts correctly is immediate from the $k \ge i$ and $j \ge k$ components of this predicate. To check that r lifts correctly, one proceeds by routine case analysis:

- $i = 1 \implies i \wedge (k \vee j) = k \vee j \ge k$
- $j = 0 \implies i \wedge (k \vee j) = i \wedge k \le k$
- The cases where $k = 0 \vee k = 1$ are immediate.

- $k \geq i \implies i \wedge (k \vee j) = i \wedge k \leq k$
- $j \geq k \implies i \wedge (k \vee j) = i \geq k$
- $i \geq k \geq j \implies i \wedge (k \vee j) = i \wedge k \leq k$

It remains to argue that $\{(i,j,k) : \Delta^2 \times \mathbb{I} \mid \Phi(i,j,k)\} \longrightarrow \Delta^2 \times \mathbb{I}$ is inner anodyne. To this end, first observe that $\{(i,j,k) : \Delta^2 \times \mathbb{I} \mid i = 1 \vee j = 0 \vee k = 0 \vee k = 1\} \longrightarrow \Delta^2 \times \mathbb{I}$ is inner anodyne by definition as the Leibniz tensor of the Segal map $\Lambda^2_1 \longrightarrow \Delta^2$ and the monomorphism $\mathrm{Bool} \to \mathbb{I}$.

Next, consider the following pushout:

$$\begin{array}{ccc}
\{(a,b) : \Delta^2 \mid a = 1 \vee b = 0\} & \longrightarrow & \left\{(i,j,k) : \Delta^2 \times \mathbb{I} \;\middle|\; \begin{array}{c} i=1 \vee j=0 \\ \vee k=0 \vee k=1 \end{array}\right\} \\
\downarrow & & \downarrow \\
\Delta^2 & \xrightarrow[(a,b) \mapsto (a,b,b)]{} & \left\{(i,j,k) : \ulcorner\Delta^2 \times \mathbb{I} \;\middle|\; \begin{array}{c} j=k \\ \vee i=1 \vee j=0 \\ \vee k=0 \vee k=1 \end{array}\right\}
\end{array}$$

By the case of 3-for-2 available for anodyne maps, we conclude that the following is anodyne:

$$\{(i,j,k) : \Delta^2 \times \mathbb{I} \mid j = k \vee i = 1 \vee j = 0 \vee k = 0 \vee k = 1\} \to \Delta^2 \times \mathbb{I}$$

We continue to glue on additional simplices to weaken the condition. Next, we consider the following:

$$\begin{array}{ccc}
\left\{(a,b,c) : \Delta^3 \;\middle|\; \begin{array}{c} a=1 \\ \vee b=c \\ \vee c=0 \end{array}\right\} & \longrightarrow & \left\{(i,j,k) : \Delta^2 \times \mathbb{I} \;\middle|\; \begin{array}{c} j=k \\ \vee i=1 \vee j=0 \\ \vee k=0 \vee k=1 \end{array}\right\} \\
\downarrow & & \downarrow \\
\Delta^3 & \xrightarrow[(a,b,c) \mapsto (a,b,c)]{} & \left\{(i,j,k) : \ulcorner\Delta^2 \times \mathbb{I} \;\middle|\; \begin{array}{c} j \geq k \\ \vee i=1 \vee j=0 \\ \vee k=0 \vee k=1 \end{array}\right\}
\end{array}$$

Again using cancellation, we conclude that the following map is Segal anodyne:

$$\{(i,j,k) : \Delta^2 \times \mathbb{I} \mid j \geq k \vee i = 1 \vee j = 0 \vee k = 0 \vee k = 1\} \to \Delta^2 \times \mathbb{I}$$

Our last goal is to allow for the tetrahedron $i \geq k$. Before gluing this on, we add an additional missing face:

$$\begin{array}{ccc}
\left\{(a,b,c) : \Delta^3 \;\middle|\; \begin{array}{c} a=1 \\ \vee b=c \\ \vee c=0 \end{array}\right\} & \longrightarrow & \left\{(i,j,k) : \Delta^2 \times \mathbb{I} \;\middle|\; \begin{array}{c} j \geq k \\ \vee i=1 \vee j=0 \\ \vee k=0 \vee k=1 \end{array}\right\} \\
\downarrow & & \downarrow \\
\Delta^3 & \xrightarrow[(a,b,c) \mapsto (a,c,b)]{} & \left\{(i,j,k) : \ulcorner\Delta^2 \times \mathbb{I} \;\middle|\; \begin{array}{c} i \geq k \geq j \vee j \geq k \\ \vee i=1 \vee j=0 \\ \vee k=0 \vee k=1 \end{array}\right\}
\end{array}$$

Finally, we consider the following pushout

$$\begin{array}{ccc} \left\{(a,b,c) : \Delta^3 \;\middle|\; \begin{smallmatrix} a=1 \\ \vee b=c \\ \vee c=0 \end{smallmatrix}\right\} & \longrightarrow & \left\{(i,j,k) : \Delta^2 \times \mathbb{I} \;\middle|\; \begin{smallmatrix} i\geq k\geq j \vee j\geq k \\ \vee i=1 \vee j=0 \\ \vee k=0 \vee k=1 \end{smallmatrix}\right\} \\ \downarrow & & \downarrow \\ \Delta^3 & \xrightarrow[(a,b,c)\mapsto(b,c,a)]{} & \left\{(i,j,k) : \Delta^{2\ulcorner}\times\mathbb{I} \;\middle|\; \begin{smallmatrix} k\geq i \vee i\geq k\geq j \vee j\geq k \\ \vee i=1 \vee j=0 \\ \vee k=0 \vee k=1 \end{smallmatrix}\right\} \end{array}$$

The result then follows from another application of cancellation/the valid case of 3-for-2. □

3 Triangulated types are hyper-triangulated

We now show that triangulated types are orthogonal to triangulations of higher-dimensional cubes. In particular, let us define the following predicate on $\mathbb{I}^n$:

$$\text{Ordered}(v) = \forall i, j < n.\, v_i \leq v_j \vee v_j \leq v_i$$

We call a type *hyper-triangulated* if it is right orthogonal to all maps $\{v : \mathbb{I}^n \mid \text{Ordered}(v)\} \to \mathbb{I}^n$. We apologize for the poor choice of name and offer the consolation that we will presently show it to be equivalent to being triangulated.

In what follows, we will want to perform induction and analysis on Φ in $\{v : \mathbb{I}^n \mid \Phi(v)\}$. We therefore define a formal grammar of such predicates amenable to such intensional analysis.

Definition 3.1. A convex predicate on $\mathbb{I}^n$ is a list of pairs $(i, j) \in \{0 \ldots n-1\}$ and denotes the conjunction of predicates $v \mapsto v_i \geq v_j$.

Definition 3.2. A tope is a list of convex predicate and denotes the disjunction of convex predicates.

Notation 3.3. In what follows, we write Φ,Ψ for topes (regarding a convex predicate as a tope silently) and write $[\![\Phi]\!]$ for the induced map $\mathbb{I}^n \to \mathsf{HProp}$. We further write $\widetilde{[\![\Phi]\!]}$ for $\{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\}$.

When Φ is convex, we write $\Phi \vdash i \leq j$ to signify that $v_i \leq v_j$ is contained in Φ and $\Phi \vdash i \leq^* j$ to signify that applying reflexivity and transitivity to those inequalities contained in Φ forces $v_i \leq v_j$ when $\Phi(v)$. It is routine to check that it is decidable whether $\Phi \vdash i \leq^* j$. We say that Φ is acyclic when $\Phi \vdash i \leq^* j, j \leq^* i$ does not hold unless $i = j$.

Lemma 3.4. *If Φ and Ψ are topes, there is a tope $\Phi \wedge \Psi$ such that $[\![\Phi]\!] \wedge [\![\Psi]\!] = [\![\Phi \wedge \Psi]\!]$. If Φ and Ψ are convex, so too is $\Phi \wedge \Psi$.*

Lemma 3.5. *Given Φ an acyclic convex tope, suppose that $r_0 : \mathbb{I}^{n-1} \to \{v : \mathbb{I}^{n-1} \mid [\![\Phi']\!](v)\}$ is a retraction to the canonical inclusion where $[\![\Phi']\!]$ is the evident induced convex tope which removes all mentions of v_{n-1} from Φ. There exists a retraction $r : \mathbb{I}^n \to \{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\}$ such that*

- *$\pi_{n-1} \circ r = \pi_{n-1}$ and,*
- *$\pi_i \circ r = \pi_i \circ r_0$ hold.*

Proof. We shall define the behavior of r component-wise (written using co-pattern notation):

$$\pi_i(r(v)) = \begin{cases} v_{n-1} & i = n-1 \\ v_{n-1} \wedge \pi_i(r(v)) & \Phi \vdash i \leq^* n-1 \\ v_{n-1} \vee \pi_i(r(v)) & \Phi(v) \vdash n-1 \leq^* i \\ \pi_i(r_0(v)) & \text{otherwise} \end{cases}$$

Here by abuse of notation we have written $r_0(v)$ for the map $\mathbb{I}^n \to \mathbb{I}^n$ which applies r_0 to the first $n-1$ components and preserves the final element unchanged. First, we note that this is well-defined as a map $\mathbb{I}^n \to \mathbb{I}^n$ since $\Phi \vdash i \leq j$ is decidable.

We must show that $[\![\Phi]\!](r(v))$ holds. Suppose that $\Phi \vdash i \leq j$ holds where either i or j is $n-1$. By the first three clauses, we immediately see that $r(v)_i \leq r(v)_j$ holds. It remains to show that if $i, j \neq n-1$ and $i \neq j$ then $r(v)_i \leq r(v)_j$ holds. We proceed by case analysis.

- If $\Phi \vdash n-1 \leq^* i$ then $r(v)_i = r_0(v)_i \vee v_{n-1} \leq r_0(v)_j \vee v_{n-1} = r(v)_j$.
- Symmetric reasoning to the above handles the case where $\Phi \vdash j \leq^* n-1$.
- If $\Phi \vdash n-1 \leq^* i$ and $\Phi \vdash j \leq^* n-1$ then we obtain a contradiction: $\Phi \vdash j \leq^* i$ yet $i \neq j$.
- If $\Phi \vdash i \leq^* n-1$ and either $\Phi \vdash n-1 \leq^* j$ or $n-1$ and j are incomparable, $r(i) = r_0(v)_i \wedge v_{n-1} \leq r_0(v)_i \leq r(v)_j$ and the conclusion follows.
- Symmetric reasoning applies if $\Phi \vdash n-1 \leq^* j$ and i and $n-1$ are incomparable.
- If i and j are both incomparable to $n-1$, the conclusion follows from our assumption on r_0. □

Let us note that the evident generalization of this lemma where rather than specifically $n-1$ we allow for an arbitrary i holds; one simply applies any permutation of $\mathbb{I}^n$ which sends the $n-1$ component to i.

Lemma 3.6. *Suppose that Φ is as above and i and j are incomparable according to Φ, then there exists a retraction $r : \mathbb{I}^n \to \{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\}$ which fixes both the i and j components.*

Proof. Choose any retraction $r_0 : \mathbb{I}^{n-2} \to \{v : \mathbb{I}^{n-2} \mid [\![\Phi'']\!](v)\}$ where Φ'' removes any mention of v_i, v_j (for instance, using the above lemma inductively). Using the above lemma, extend this retraction to $r_1 : \mathbb{I}^{n-1} \to \{v : \mathbb{I}^{n-1} \mid [\![\Phi']\!](v)\}$ which Φ' only removes mentions of v_j. By the above lemma, we know that $r_1(v)_i = v_i$.

Next, extend this retraction to Φ again using the above lemma $r : \mathbb{I}^n \to \{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\}$ and, since we have assumed Φ leaves v_i and v_j are incomparable, v_i continues to be fixed by r. □

Lemma 3.7. *Suppose that Φ is as above and further suppose that i and j are incomparable according to Φ. Then the following map is triangulated anodyne:*

$$\{v : \mathbb{I}^n \mid [\![\Phi]\!](v) \wedge (v_i \leq v_j \vee v_j \leq v_i)\} \longrightarrow \{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\}$$

Proof. Using the above lemma, choose a retraction $\mathbb{I}^n \to \{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\}$ which fixes v_i and v_j. The conclusion then follows from the section-retraction diagram:

$$\begin{array}{ccccc}
\left\{v : \mathbb{I}^n \;\middle|\; [\![\Phi]\!](v) \wedge {\scriptstyle \vee v_j \leq v_i}^{\,v_i \leq v_j}\right\} & \longrightarrow & \{v : \mathbb{I}^n \mid v_i \leq v_j \vee v_j \leq v_i\} & \longrightarrow & \{v : \mathbb{I}^n \mid [\![\Phi]\!](v) \wedge {\scriptstyle \vee v_j \leq v_i}^{\,v_i \leq v_j}\} \\
\downarrow & & \downarrow & & \downarrow \\
\{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\} & \longrightarrow & \mathbb{I}^n & \longrightarrow & \{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\}
\end{array}$$

In particular, since the retraction fixes v_i and v_j it correctly lifts. □

Lemma 3.8. *If a tope Φ has the property that for all convex topes Ψ the map $\{v : \mathbb{I}^n \mid [\![\Phi]\!](v) \wedge [\![\Psi]\!](v)\} \to \{v : \mathbb{I}^n \mid [\![\Psi]\!](v)\}$ is triangulated anodyne then the same holds for all topes Ψ.*

Proof. Let us induct on the number of convex components in Ψ. The base case is as the map is an isomorphism.

For the inductive step, suppose that $\widetilde{[\![\Phi]\!]} \cap \widetilde{[\![\Psi]\!]} \hookrightarrow \widetilde{[\![\Psi]\!]}$ is triangulated anodyne whenever Ψ can be written as the union of k convex components. Suppose now that $\Psi = \Psi_1 \vee \Psi_2$ where Ψ_1 is k convex

topes and Ψ_2 is convex. From the induction hypothesis it follows that both $\widetilde{[\![\Psi_1]\!]} \cap \widetilde{[\![\Phi]\!]} \hookrightarrow \widetilde{[\![\Psi_1]\!]}$ and $\widetilde{[\![\Psi_2]\!]} \cap \widetilde{[\![\Phi]\!]} \hookrightarrow \widetilde{[\![\Psi_2]\!]}$ are triangulated anodyne.

Moreover, note that distributivity allows us to write $\widetilde{[\![\Psi_1]\!]} \cap \widetilde{[\![\Psi_2]\!]}$ as a tope using at most k convex components and hence the map the following map is triangulated anodyne:

$$(\widetilde{[\![\Psi_1]\!]} \cap \widetilde{[\![\Psi_2]\!]}) \cap \widetilde{[\![\Phi]\!]} \cong (\widetilde{[\![\Psi_1]\!]} \cap \widetilde{[\![\Phi]\!]}) \cap (\widetilde{[\![\Psi_2]\!]} \cap \widetilde{[\![\Phi]\!]}) \hookrightarrow \widetilde{[\![\Psi_1]\!]} \cap \widetilde{[\![\Psi_2]\!]}$$

It follows that $\widetilde{[\![\Psi]\!]} \cap \widetilde{[\![\Phi]\!]} \hookrightarrow \widetilde{[\![\Psi]\!]}$ is triangulated anodyne as it is the pushout of the canonical span of these three maps. □

Lemma 3.9. *For any tope Φ the map following map is triangulated anodyne:*

$$\{v : \mathbb{I}^n \mid [\![\Phi]\!](v) \wedge (v_i \leq v_j \vee v_j \leq v_i)\} \longrightarrow \{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\}$$

Proof. By the above lemma, it suffices to show that this is the case whenever $\{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\}$ is convex. Note that if Φ is convex but not acyclic, $[\![\Phi]\!]$ is equivalent to an acyclic convex predicate on a smaller cube and on this smaller cube cut out by $\widetilde{[\![\Phi]\!]}$ either $v_i \leq v_j \vee v_j \leq v_i$ remains unchanged—as Φ does not for $i = j$—or becomes $\top$. As the latter case is trivial, we may safely assume that Φ is acyclic convex. If v_i and v_j are incomparable according to Φ, then Lemma 3.7 applies. On the other hand, if they are comparable then the above map is an isomorphism and so the result follows. □

Lemma 3.10. *If X is a triangulated type, it is right orthogonal to $\{v : \mathbb{I}^n \mid \mathrm{Ordered}(v)\} \to \mathbb{I}^n$.*

Proof. We generalize to showing the following map is triangulated anodyne for all topes Φ:

$$\{v : \mathbb{I}^n \mid [\![\Phi]\!](v) \wedge \mathrm{Ordered}(v)\} \longrightarrow \{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\}$$

We shall proceed by induction on the number of incomparable pairs (v_i, v_j) according to Φ. The base case where Φ proves that every pair is comparable is trivial as the above map is an isomorphism. In the inductive step, Lemma 3.9 tells us that if v_i and v_j are incomparable then the following map is triangulated anodyne:

$$\{v : \mathbb{I}^n \mid [\![\Phi]\!](v) \wedge v_i \leq v_j \vee v_j \leq v_i\} \longrightarrow \{v : \mathbb{I}^n \mid [\![\Phi]\!](v)\}$$

Applying the induction hypothesis to the tope produced by intersecting Φ with $v_i \leq v_j \vee v_j \leq v_i$ and noting that $\mathrm{Ordered}(v)$ contains $v_i \leq v_j \vee v_j \leq v_i$ as a conjunct, the conclusion follows by the stability of anodyne maps under composition. □

Alternative proof of Lemma 3.10. We record the following more direct but combinatorial proof of Lemma 3.10 for posterity.

Proof of Lemma 3.10. We proceed by a somewhat complex induction. We argue that for all n and σ a permutation of $k \leq n$ elements then the following map is triangulated anodyne:

$$\{v : \mathbb{I}^n \mid \exists \bar{\sigma} : \mathrm{Perm}(n).\, \mathrm{extends}(\bar{\sigma}, \sigma) \wedge v_{\bar{\sigma}(0)} \geq \cdots \geq v_{\bar{\sigma}(n-1)}\} \to \{v : \mathbb{I}^n \mid v_{\sigma(0)} \geq \cdots \geq v_{\sigma(k-1)}\}$$

Here $\mathrm{extends}(\bar{\sigma}, \sigma)$ is the relation stating that $\sigma(i) \leq \sigma(j)$ if and only if $\bar{\sigma}(i) \leq \bar{\sigma}(j)$.

If $k = n$, we are done. Suppose for a moment that $k + 1 = n$. For notational simplicity in what follows, we assume that σ is the identity permutation but this has no real impact. By our above lemma, the following map is triangulated anodyne:

$$\{v : \mathbb{I}^{k+1} \mid v_0 \geq \cdots \geq v_{k-1} \wedge (v_{k-1} \leq v_k \vee v_k \leq v_{k-1})\} \to \{v : \mathbb{I}^{k+1} \mid v_0 \geq \cdots \geq v_{k-1}\}$$

We can distribute the $\wedge$ over the $\vee$ to rewrite the top condition as $v_0 \geq \cdots \geq v_{k-1} \geq v_k \vee (v_{k-1} \leq v_k \vee v_k \leq v_{k-1} \wedge v_k \geq v_{k-1})$. Moreover, arguing inductively and using the closure of anodyne maps under pushouts, we can see that the following map is triangulated anodyne:

$$\{v : \mathbb{I}^{k+1} \mid \exists 0 \leq i \leq k.\, v_0 \geq \cdots \geq v_{i-1} \geq v_k \geq v_i \geq \cdots \geq v_{k-1}\} \to \{v : \mathbb{I}^{k+1} \mid v_0 \geq \cdots \geq v_{k-1}\}$$

Here we by convention take $v_{-1} = 1$ and $v_{k+1} = 0$. This argument concludes the case where $k+1 = n$.

For the general case, the above reasoning tells us that it suffices to argue that the following map is triangulated anodyne:

$$\{v : \mathbb{I}^n \mid \exists \bar{\sigma} : \mathsf{Perm}(n).\, \mathsf{extends}(\bar{\sigma}, \sigma) \wedge v_{\bar{\sigma}(0)} \geq \cdots \geq v_{\bar{\sigma}(n-1)}\} \\ \to \{v : \mathbb{I}^n \mid \exists 0 \leq i \leq k.\, v_0 \geq \cdots \geq v_{i-1} \geq v_k \geq v_i \geq \cdots \geq v_{k-1}\}$$

Pulling back this map along the canonical inclusion of $\{v : \mathbb{I}^n \mid \cdots \geq v_{i-1} \geq v_k \geq v_i \geq \dots\}$ for each i is triangulated anodyne by our induction hypothesis. Moreover, pulling back this map along any intersection of these conditions (say for $i_0 < \dots < i_j$) is equivalent to the following map

$$\{v : \mathbb{I}^n \mid \exists \bar{\sigma} : \mathsf{Perm}(n - (i_j - i_0 + 1)).\, \mathsf{extends}(\bar{\sigma}, \mathsf{id}) \wedge v_{\bar{\sigma}(0)} \geq \cdots \geq v_{\bar{\sigma}(n-1)}\} \\ \to \{v : \mathbb{I}^{n-(i_j - i_0+1)} \mid v_{i-1} \geq v_k \geq v_i \geq \cdots \geq v_{k-1}\}$$

This is in turn is triangulated anodyne by our induction hypothesis. As a result, the union of all these maps is triangulated anodyne, completing our proof. □

4 ♭-triangulated types are simplicial

The next step in the argument is to show that all ♭-annotated triangulated types are simplicial. The ♭-annotation is used only in the following arguments:

Lemma 4.1. *Fix $X :_\flat \{v : \mathbb{I}^n \mid \mathrm{Ordered}(v)\} \to \mathcal{U}$ such that for any map $i :_\flat \Delta^m \to \mathbb{I}^n$ the following holds*

$$\mathsf{isContr}(\langle \flat \mid (v : \Delta^m) \to X(i\,v)\rangle)$$

The type $\langle \flat \mid \{v : \mathbb{I}^n \mid \mathrm{Ordered}(v)\} \to X(v)\rangle$ is then contractible.

Proof. We prove that the slightly more general claim that the claim holds for any $\phi = \phi_1 \vee \cdots \vee \phi_m : \mathbb{I}^n \to \mathsf{HProp}$ such that $\{v : \mathbb{I}^n \mid \phi_i\} \cong \Delta^n$. We quickly note that given such a ϕ, $\phi_1 \wedge \phi_0 \vee \dots \phi_m \wedge \phi_0$ also satisfies this condition: the intersection of two total orderings on $\mathbb{I}^n$ is a total ordering on a (possibly smaller) $\mathbb{I}^m$.

We proceed by induction on m. The cases where $m = 0, 1$ are immediate. If $m \geq 2$ then we first note that by converting a map out of a disjunction into a pullback and leveraging the fact that $\langle \flat \mid - \rangle$ preserves limits, we then find the following:

$$\langle \flat \mid \{v : \mathbb{I}^n \mid \phi(v)\} \to X(v)\rangle \\ \simeq \langle \flat \mid \{v : \mathbb{I}^n \mid \phi_1(v)\} \to X(v)\rangle \times_{\langle \flat \mid \{v : \mathbb{I}^n \mid \phi_1(v) \wedge \phi_2(v) \vee \dots\} \to X(v)\rangle} \langle \flat \mid \{v : \mathbb{I}^n \mid \phi_2(v) \vee \dots\} \to X(v)\rangle$$

All three of these types are contractible: the first by our assumption since $\{v \mid \phi_1(v)\}$ is isomorphic to a simplex and the latter two by our induction hypothesis. The conclusion then follows as the limit of contractible types is contractible. □

Lemma 4.2. *If $X, Y :_\flat \mathcal{U}$ are ♭-orthogonal to $\{v : \mathbb{I}^n \mid \mathrm{Ordered}(v)\} \to \mathbb{I}^n$ then a map $f :_\flat X \to Y$ is an equivalence if and only if for all $n : \mathsf{Nat}$, the following holds*

$$\mathsf{isEquiv}((f_*)^\dagger : \langle \flat \mid \Delta^n \to X\rangle \to \langle \flat \mid \Delta^n \to Y\rangle)$$

Remark 4.3. Here by ♭-orthogonal we mean that the comparison map $X^{\mathbb{I}^n} \to X^{\{v : \mathbb{I}^n \mid \dots\}}$ is an equivalence after applying $\langle \flat \mid - \rangle$ to both sides.

Proof. The only if direction is clear, so we focus on the if. By the cubes-separate axiom, it suffices to show that the following maps are equivalences:

$$(f_*)^\dagger : \langle \flat \mid \mathbb{I}^n \to X\rangle \to \langle \flat \mid \mathbb{I}^n \to Y\rangle$$

Moreover, since X and Y are orthogonal to $\{v : \mathbb{I}^n \mid \mathrm{Ordered}(v)\} \to \mathbb{I}^n$, this in turn is equivalent to the following condition:

$$\mathsf{isEquiv}((f_*)^\dagger : \langle \flat \mid \{v : \mathbb{I}^n \mid \mathrm{Ordered}(v)\} \to X\rangle \to \langle \flat \mid \{v : \mathbb{I}^n \mid \mathrm{Ordered}(v)\} \to Y\rangle)$$

We must show that if $y :_\flat \{v : \mathbb{I}^n \mid \mathrm{Ordered}(v)\} \to Y$ then $\langle \flat \mid \{v : \mathbb{I}^n \mid \mathrm{Ordered}(v)\} \to X_{y(v)}\rangle$—the fiber of the above map—is contractible. Our assumption is that the restriction of this type along any map $\Delta^m \to \mathbb{I}^n$ is contractible and so the conclusion follows by Lemma 4.1. □

Lemma 4.4. *A type $X :_\flat \mathcal{U}$ is simplicial iff it is right $\flat$-orthogonal to $\{v : \mathbb{I}^n \mid \mathrm{Ordered}(v)\} \to \mathbb{I}^n$.*

Proof. The only if is clear: $\{v : \mathbb{I}^n \mid \mathrm{Ordered}\} \longrightarrow \mathbb{I}^n$ is an intersection of simplicial-anodyne maps and, as a left-exact localization, simplicial anodyne maps are stable under intersections (indeed, under arbitrary pullback).

For the if direction, we must show the unit map $X \to \boxslash X$ is an equivalence. As both sides are $\flat$-orthogonal $\{v : \mathbb{I}^n \mid \mathrm{Ordered}(v)\} \longrightarrow \mathbb{I}^n$ it suffices to argue that the following holds:

$$\mathsf{isEquiv}((\eta_*)^\dagger : \langle \flat \mid \Delta^n \to X\rangle \to \langle \flat \mid \Delta^n \to \boxslash X\rangle)$$

This, in turn, is precisely the simplicial stability axiom. □

We now arrive at our main result.

Theorem 1.4. *If $X :_\flat \mathcal{U}$ is Segal then X is simplicial.*

Proof. Combine Lemma 2.2, Lemma 3.10, and Lemma 4.4. Note that in the last lemma we require that X be $\flat$-annotated. □

A The simplicial stability axiom

In this appendix, we give a proof of the simplicial stability axiom from the assumption that $\mathbb{I}$ is tiny and satisfies duality, two other axioms of $\mathsf{TT}_\boxslash$. Both of these axioms are, in some sense, more primitive than Axiom 4: duality is a fundamental feature of the classifying topos of bounded distributive lattices (of which both $\mathrm{PSh}(\square)$ and $\mathrm{PSh}(\Delta)$ are subtopoi) and likewise the tinyness of $\mathbb{I}$ ensures that it is interpreted by a representable presheaf. Moreover, both axioms are used in several places within the development of simplicial type theory. In contrast, simplicial stability is a specific statement about $\boxslash$, for which we have both a universal property available, and used only to prove that $\mathcal{S}$ and Cat are simplicial. In light of the main result of this paper, these arguments can be removed and this appendix further shows that simplicial stability can go with them.

Remark A.1. The contents of this appendix were first produced in conjunction with an LLM. While this process resulted in an autoformalization relative to the `agda-synthetic-categories` library, the resulting code obscured the essential simplicity of the argument. The following proof is a "de-formalization" of that Agda code.

First, we shall need to use the tinyness of $\mathbb{I}$ to show the following:

Lemma A.2. *For any n, Δ^n is tiny. That is, the following proposition holds under $\flat$ for any $A :_\flat \mathcal{U}$:*

$$\textstyle\sum_{A_{\Delta^n} : \mathcal{U}, \epsilon : (A_{\Delta^n})^{\Delta^n} \to A} \prod_{B :_\flat \mathcal{U}} \mathsf{isEquiv}(\lambda \mathsf{mod}_\flat(f).\, \mathsf{mod}_\flat(\epsilon \circ f_*) : \langle \flat \mid B \to A_{\Delta^n}\rangle \to \langle \flat \mid B^{\Delta^n} \to A\rangle)$$

Proof. Since $\mathbb{I}$ is tiny, so too is $\mathbb{I}^n$. It suffices to show that tiny objects are closed under retracts as Δ^n is a retract of $\mathbb{I}^n$. To show this, we essentially argue that if C is tiny and we are given a (quasi)-idempotent $e : C \to C$, then the amazing right adjoint A_C inherits a quasi-idempotent $\hat{e} : A_C \to A_C$ by transposition. Splitting this quasi-idempotent following Shulman [Shu17] is easily seen to give rise to the correct amazing right adjoint.

The main difficulty here is in arguing that we indeed obtain an quasi-idempotent $\hat{e} : A_C \to A_C$. First, we choose a (unique) $\hat{e}$ and a path $p : \epsilon \circ \hat{e}_* =_{(C \to A_C) \to A} \epsilon \circ e^*$. Next, we lift the homotopy $I : e \circ e = e$ to $\hat{e}$ by defining $\hat{I}$ as the unique homotopy such that $\mathsf{ap}_{\epsilon \circ (-)_*} \hat{I}$ is given as follows:

$$\mathsf{ap}_{\epsilon \circ (-)_*} \hat{I} = \mathsf{ap}_{- \circ \hat{e}_*} p \bullet \mathsf{ap}_{- \circ e^*} p \bullet \mathsf{ap}_{\epsilon \circ - ^*} I \bullet p^{-1}$$

Finally, we must show that this is a quasi-idempotent. That is, we must produce an identification $I \circ \hat{e} = \hat{e} \circ I$. It suffices to show this after the application of $\mathsf{ap}_{\epsilon \circ (-)_*}$:

$$\begin{aligned}
&\mathsf{ap}_{\epsilon \circ (-)_*} (\hat{I} \circ \hat{e}) \\
&= \mathsf{ap}_{- \circ \hat{e}_*} \mathsf{ap}_{\epsilon \circ (-)_*} \hat{I} \\
&= \mathsf{ap}_{- \circ \hat{e}_* \circ \hat{e}_*} p \bullet \mathsf{ap}_{- \circ e^* \circ \hat{e}_*} p \bullet \mathsf{ap}_{\epsilon \circ - ^* \circ \hat{e}_*} I \bullet \mathsf{ap}_{- \circ \hat{e}_*} p^{-1} \\
&= \mathsf{ap}_{- \circ \hat{e}_* \circ \hat{e}_*} p \bullet \mathsf{ap}_{- \circ \hat{e}_* \circ e^*} p \bullet \mathsf{ap}_{- \circ e^* \circ e^*} p \bullet \mathsf{ap}_{\epsilon \circ e^* \circ - ^*} I \bullet \mathsf{ap}_{- \circ e^*} p^{-1} \bullet \mathsf{ap}_{- \circ \hat{e}_* \circ \hat{e}_*} p^{-1} \\
&= \mathsf{ap}_{- \circ \hat{e}_* \circ \hat{e}_*} p \bullet \mathsf{ap}_{- \circ \hat{e}_* \circ e^*} p \bullet \mathsf{ap}_{- \circ e^* \circ e^*} p \bullet \mathsf{ap}_{\epsilon \circ - ^* \circ e^*} I \bullet \mathsf{ap}_{- \circ e^*} p^{-1} \bullet \mathsf{ap}_{- \circ \hat{e}_* \circ \hat{e}_*} p^{-1} \qquad \dagger \\
&= \mathsf{ap}_{- \circ \hat{e}_* \circ \hat{e}_*} p \bullet \mathsf{ap}_{- \circ e^*} \mathsf{ap}_{\epsilon \circ (-)_*} \hat{I} \bullet \mathsf{ap}_{- \circ \hat{e}_* \circ \hat{e}_*} p^{-1} \\
&= \mathsf{ap}_{- \circ \hat{e}_* \circ \hat{e}_*} p \bullet \mathsf{ap}_{\epsilon \circ (-)_* \circ e^*} \hat{I} \bullet \mathsf{ap}_{- \circ \hat{e}_* \circ \hat{e}_*} p^{-1} \\
&= \mathsf{ap}_{\epsilon \circ \hat{e}_* \circ (-)_*} \hat{I} \\
&= \mathsf{ap}_{\epsilon \circ (-)_*} (\hat{e} \circ \hat{I})
\end{aligned}$$

Here † uses the quasi-idempotence of I and the remaining steps are formal. □

We now restate our goal as follows. Fix $X :_\flat \mathcal{U}$. We must show that for all $Y :_\flat \mathcal{U}$ that the following map is an equivalence:

$$(((\eta_*)^\dagger)^*)^\dagger : \langle \flat \mid \langle \flat \mid \Delta^n \to \between X \rangle \to Y \rangle \to \langle \flat \mid \langle \flat \mid \Delta^n \to X \rangle \to Y \rangle$$

To this end, we note that there is a commuting square of the following shape:

$$\begin{array}{ccc}
\langle \flat \mid \langle \flat \mid \Delta^n \to \between X \rangle \to Y \rangle & \xrightarrow{(((\eta_*)^\dagger)^*)^\dagger} & \langle \flat \mid \langle \flat \mid \Delta^n \to X \rangle \to Y \rangle \\
\downarrow & & \downarrow \\
\langle \flat \mid \between X \to \langle \sharp \mid Y \rangle_{\Delta^n} \rangle & \xrightarrow[(\eta^*)^\dagger]{} & \langle \flat \mid X \to \langle \sharp \mid Y \rangle_{\Delta^n} \rangle
\end{array}$$

The vertical maps (given by transposition) are equivalences, so the top map is an equivalence if and only if the bottom map is as well. We now note that the bottom map is an equivalence if $\langle \sharp \mid Y \rangle_{\Delta^n}$ is simplicial, which we now prove.

Lemma A.3. *If $Y :_\flat \mathcal{U}$ then $\langle \sharp \mid Y \rangle_{\Delta^n}$ is simplicial.*

Proof. Fix $Z :_\flat \mathcal{U}$. We shall show that the following map is an equivalence:

$$\left\langle \flat \mid Z \to \textstyle\sum_{i,j:\mathbb{I}} \langle \sharp \mid Y \rangle_{\Delta^n} \right\rangle \to \left\langle \flat \mid Z \to \textstyle\sum_{i,j:\mathbb{I}} i \le j \vee j \le i \to \langle \sharp \mid Y \rangle_{\Delta^n} \right\rangle$$

Passing to fibers, we may suppose that that $f, g :_\flat Z \to \mathbb{I}$. We must show that the following canonical map is an equivalence:

$$\langle \flat \mid Z \to \langle \sharp \mid Y \rangle_{\Delta^n} \rangle \to \left\langle \flat \mid \textstyle\sum_{z:Z} f\, z \le g\, z \vee g\, z \le f\, z \to \langle \sharp \mid Y \rangle_{\Delta^n} \right\rangle$$

Transposing along the $\flat \dashv \sharp$ and the amazing adjunction, we must show that the following canonical map is an equivalence:

$$\left\langle \flat \mid \textstyle\sum_{z : Z^{\Delta^n}} f \circ z \leq g \circ z \vee g \circ z \leq f \circ z \right\rangle \to \langle \flat \mid Z^{\Delta^n} \rangle$$

However, in this context we observe that $f \circ z, g \circ z$ are $\flat$-annotated maps $\Delta^n \to \mathbb{I}$. By duality, this type is equivalent to $\mathbb{I}[x_1 \geq \cdots \geq x_n]$ whose $\flat$-elements are totally ordered. Consequently, $f \circ z \leq g \circ z \vee g \circ z \leq f \circ z$ always holds in the above context and the conclusion follows. □

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