Normalization for multimodal type theory

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To a first approximation, type theory is the study of objects invariant under change of context i.e. fibered connectives. Unfortunately, many important features of models of type theory are not fibered and this limits the utility of type theory as an internal language. To manage this contradiction, a number of modal type theories have been proposed which allow for a controlled introduction of non-fibered connectives. Such modal type theories are often delicate to construct, which has motivated type theories which can be instantiated to different modal situations [10, 14].

We focus on one such general modal type theory: MTT, a multimodal dependent type theory [10]. MTT can be instantiated by a strict 2-category specifying a collection of modes, modalities and natural transformations between them—a mode theory—and by altering the input mode theory MTT can be used to reason about e.g., guarded recursion, internalized parametricity, axiomatic cohesion, and the metatheory of type theories [6]. Each instantiation of MTT is known to enjoy certain metatheorems regardless of the mode theory (soundness, canonicity) however, lacking a similarly general normalization result, MTT cannot be implemented in a way which permits easy reuse between different mode theories.

We contribute a normalization algorithm for MTT which applies irrespective of its mode theory [9]. As a corollary of this normalization result, we show that type constructors are injective and that conversion in MTT is decidable if and only if equality in the underlying mode theory is decidable. As a further consequence, we show that the typechecking problem for MTT is decidable under these same circumstances. Accordingly, this result is provides a theoretical grounding for an implementation which can be instantiated to a variety of modal situations.

Normalization by gluing Our normalization proof follows in the tradition of semantic proofs of normalization and gluing arguments [1–3, 7, 8, 13, 16, 20]. More precisely, we do not proceed by fixing rewriting system which we prove to be strongly normalizing nor by defining a normalization algorithm on raw terms which is then shown to be sound and complete. Instead, we construct a model in the category given by gluing the syntactic model along a nerve restricting from substitutions to renamings. Normalization follows from this model together with the initiality of syntax. While this approach is standard for normalization-by-gluing proofs, the multimodal apparatus introduces several complications.

MTT Cosmoi In order to discuss the novel features of this proof, we must review the model theory of MTT. For the remainder of this abstract, we consider MTT over a fixed mode theory $\mathcal{M}$. A model of MTT is a strict 2-functor $F : \mathcal{M}^{\mathsf{coop}} \longrightarrow \mathbf{Cat}$ sending $m : \mathcal{M}$ to a category $F(m)$ which supports a model of MLTT. More precisely, in the language of natural models, we require a representable natural transformation $\tau_m : \dot{\mathcal{T}}_m \longrightarrow \mathcal{T}_m$ in $\mathbf{PSh}(F(m))$ closed under the standard connectives of type theory [4]. For each modality $\mu : n \longrightarrow m$, we require a commutative square:

$$\begin{array}{ccc} F(\mu)^*\dot{\mathcal{T}}_n & \longrightarrow & \dot{\mathcal{T}}_m \\ \downarrow & & \downarrow \\ F(\mu)^*\mathcal{T}_n & \longrightarrow & \mathcal{T}_m \end{array}$$

The bottom (resp. top) map of this commutative square interprets the formation (resp. introduction) rule of the modal type, see Gratzer et al. [11, Section 5] for further details.

The extra complexity of a 2-functor of models precludes directly constructing the necessary model of MTT. Instead, we begin by generalizing the definition of models following Gratzer and Sterling [12]. We require a pseudofunctor $G : \mathcal{M} \longrightarrow \mathbf{Cat}$ such that $G(m)$ is merely locally Cartesian closed and $G(\mu)$ is a right adjoint, intuitively capturing $\mathbf{PSh}(F(m))$ and $F(m)^*$ respectively. Even in this weaker setting, we can still state all the requirements of a model (a universe τ_m closed under various connectives, etc.) with one exception: we must drop the requirement that each τ_m is fiberwise representable. A pseudofunctor equipped with the remaining applicable structure is an *MTT cosmos*. Note that a model of MTT induces a cosmos, in particular presheaves over contexts induce a cosmos $\mathcal{S}[\![-]\!]$.

We define a category of renamings Ren_m and a functor $\mathbf{i}[m]$ embedding it into the category of contexts Cx_m. Similarly, we define neutral and normal forms together with an embedding into terms such that normal forms correspond to β-normal and η-long terms and show that they form a presheaves over Ren_m. Like terms, normal forms contain modalities and 2-cells from $\mathcal{M}$ so, while they are not quotiented by a definitional equality, their equality is decidable if and only if the equality in $\mathcal{M}$ is decidable.

By gluing along $\mathbf{i}[m]^* : \mathbf{PSh}(\mathsf{Cx}_m) \longrightarrow \mathbf{PSh}(\mathsf{Ren}_m)$, we obtain a presheaf topos $\mathcal{G}[\![m]\!]$ and the 2-naturality of $\mathbf{i}[-]$ ensures that $\mathcal{G}[\![-]\!]$ organizes into a 2-functor out of $\mathcal{M}$. The crux of our normalization argument is the construction of a cosmos in $\mathcal{G}[\![-]\!]$ lying over $\mathcal{S}[\![-]\!]$.

The normalization cosmos While $\mathcal{G}[\![m]\!]$ is a presheaf topos, it is cumbersome to manipulate directly. In order to construct the normalization cosmos in $\mathcal{G}[\![-]\!]$, we adapt synthetic Tait computability (STC) [17–19] to our multimodal setting and work exclusively in the internal language of $\mathcal{G}[\![-]\!]$. Specifically, we show that MTT can be interpreted into $\mathcal{G}[\![-]\!]$ and under this interpretation there is a proposition $\mathbf{syn}_m$ which presents $\mathcal{S}[\![m]\!] = \mathbf{PSh}(\mathsf{Cx}_m)$ (resp. $\mathbf{PSh}(\mathsf{Ren}_m)$) as an open (resp. closed) subtopos of $\mathcal{G}[\![m]\!]$. Having relaxed from models of MTT to cosmoi, the required additional structure can be presented as a sequence of constants to be implemented in the internal language, with the open modality being used to ensure that a connective lies strictly over its counterpart in $\mathcal{S}[\![-]\!]$.

Working internally, we substantiate the constants of an MTT cosmos while ensuring that they lie strictly over their counterparts in $\mathcal{S}[\![m]\!]$. Unlike typical gluing proofs, there is no need to ever exit the internal language and so many subtle constructions are transformed into a sequence of programming exercises in MTT. Following Sterling and Harper [19], we use the internal realignment axiom [5, 15] to ensure that natural constructions various connectives lie strictly over their counterparts in $\mathcal{S}[\![-]\!]$. We conclude that there is an MTT cosmos in $\mathcal{G}[\![-]\!]$ and a morphism of cosmoi $\pi : \mathcal{G}[\![-]\!] \longrightarrow \mathcal{S}[\![-]\!]$.

The normalization function While syntax $\mathcal{S}[\![-]\!]$ is the initial model of MTT, it is not initial among cosmoi. It still, however, enjoys a privileged position in this category which ensures that each context, term, and type in $\mathcal{S}[\![-]\!]$ is in the essential image of π. From this fact and the definition of π, we conclude the following almost immediately.

Theorem 1. *Each term and type in MTT has a unique normal form.*

As our proof of this fact is constructive, it yields an effective procedure which parallels a familiar normalization-by-evaluation algorithm. Consequently, we obtain the following:

Corollary 2. *The conversion problem in MTT is equivalent to the conversion problem of $\mathcal{M}$.*

Corollary 3. *If modalities and 2-cells enjoy decidable equality, typechecking MTT is decidable.*

References

[1] T. Altenkirch, P. Dybjer, M. Hofmann, and P. Scott. 2001. Normalization by Evaluation for Typed Lambda Calculus with Coproducts. In *Proceedings of the 16th Annual IEEE Symposium on Logic in Computer Science* (Washington, DC, USA) *(LICS '01)*. IEEE Computer Society, 303–. http://dl.acm.org/citation.cfm?id=871816.871869

[2] Thorsten Altenkirch, Martin Hofmann, and Thomas Streicher. [n.d.]. Categorical reconstruction of a reduction free normalization proof. In *Category Theory and Computer Science* (Berlin, Heidelberg, 1995), David Pitt, David E. Rydeheard, and Peter Johnstone (Eds.). Springer Berlin Heidelberg, 182–199.

[3] Thorsten Altenkirch and Ambrus Kaposi. 2016. Normalisation by Evaluation for Dependent Types. In *1st International Conference on Formal Structures for Computation and Deduction (FSCD 2016) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 52)*, Delia Kesner and Brigitte Pientka (Eds.). Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 6:1–6:16. https://doi.org/10.4230/LIPIcs.FSCD.2016.6

[4] Steve Awodey. 2018. Natural models of homotopy type theory. *Mathematical Structures in Computer Science* 28, 2 (2018), 241–286. https://doi.org/10.1017/S0960129516000268 arXiv:1406.3219

[5] Lars Birkedal, Aleš Bizjak, Ranald Clouston, Hans Bugge Grathwohl, Bas Spitters, and Andrea Vezzosi. 2019. Guarded Cubical Type Theory. *Journal of Automated Reasoning* 63 (2019), 211–253.

[6] Rafaël Bocquet, Ambrus Kaposi, and Christian Sattler. 2021. Induction principles for type theories, internally to presheaf categories. arXiv:2102.11649 [cs.LO]

[7] Thierry Coquand. 2019. Canonicity and normalization for dependent type theory. *Theoretical Computer Science* 777 (2019), 184–191. https://doi.org/10.1016/j.tcs.2019.01.015

[8] Marcelo Fiore. 2002. Semantic Analysis of Normalisation by Evaluation for Typed Lambda Calculus. In *Proceedings of the 4th ACM SIGPLAN International Conference on Principles and Practice of Declarative Programming* (Pittsburgh, PA, USA) *(PPDP '02)*. ACM, 26–37. https://doi.org/10.1145/571157.571161

[9] Daniel Gratzer. 2021. Normalization for multimodal type theory. In preparation. Available https://jozefg.github.io/papers/normalization-for-multimodal-type-theory.pdf.

[10] Daniel Gratzer, G.A. Kavvos, Andreas Nuyts, and Lars Birkedal. 2020. Multimodal Dependent Type Theory. In *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS '20)*. ACM. https://doi.org/10.1145/3373718.3394736

[11] Daniel Gratzer, G.A. Kavvos, Andreas Nuyts, and Lars Birkedal. 2020. Multimodal Dependent Type Theory. Extended version of the LICS paper by the same name.

[12] Daniel Gratzer and Jonathan Sterling. 2020. Syntactic categories for dependent type theory: sketching and adequacy. arXiv:2012.10783 [cs.LO]

[13] Ambrus Kaposi, Simon Huber, and Christian Sattler. 2019. Gluing for type theory. In *Proceedings of the 4th International Conference on Formal Structures for Computation and Deduction (FSCD 2019)*, Herman Geuvers (Ed.), Vol. 131.

[14] Daniel R. Licata, Michael Shulman, and Mitchell Riley. 2017. A Fibrational Framework for Substructural and Modal Logics. In *2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017) (Leibniz International Proceedings in Informatics (LIPIcs), Vol. 84)*, Dale Miller (Ed.). Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, 25:1–25:22. https://doi.org/10.4230/LIPIcs.FSCD.2017.25

[15] Ian Orton and Andrew M. Pitts. 2018. Axioms for Modelling Cubical Type Theory in a Topos. *Logical Methods in Computer Science* 14, 4 (2018). https://doi.org/10.23638/LMCS-14(4:23)2018 arXiv:1712.04864

[16] Michael Shulman. 2015. Univalence for inverse diagrams and homotopy canonicity. *Mathematical Structures in Computer Science* 25, 5 (2015), 1203–1277. https://doi.org/10.1017/S0960129514000565 arXiv:1203.3253

[17] Jonathan Sterling. 2021. First Steps in Synthetic Tait Computability. (2021). Thesis draft.

[18] Jonathan Sterling and Daniel Gratzer. 2020. Lecture Notes on Synthetic Tait Computability.

[19] Jonathan Sterling and Robert Harper. 2020. Logical Relations As Types: Proof-Relevant Parametricity for Program Modules. (2020). arXiv:2010.08599 https://arxiv.org/abs/2010.08599 Draft paper.

[20] Thomas Streicher. 1998. Categorical intuitions underlying semantic normalisation proofs. In *Preliminary Proceedings of the APPSEM Workshop on Normalisation by Evaluation*, O. Danvy and P. Dybjer (Eds.). Department of Computer Science, Aarhus University.