

Directed univalence in simplicial homotopy type theory

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Riehl and Shulman’s simplicial type theory extends homotopy type theory with a directed interval type, allowing it to be modeled in simplicial spaces (and simplicial objects in any higher topos). The main application is the development of synthetic higher category theory, modeling $(\infty, 1)$ -categories as types satisfying a complete Segal condition.

We extend simplicial type theory with modalities and new reasoning principles to obtain *triangulated type theory* in order to construct the $(\infty, 1)$ -category of spaces \mathcal{S} , and from it many other concrete $(\infty, 1)$ -categories.

We prove that homomorphisms in \mathcal{S} correspond to ordinary functions of types, i.e., that \mathcal{S} is *directed univalent*. From this we can significantly extend the reach of synthetic higher category theory with more results and examples, including the first complete examples of the *structure homomorphism principle*, a directed version of the structure identity principle known from homotopy type theory.

1 Introduction

Homotopy type theory (HoTT) is a type theory for synthetic ∞ -groupoid theory; it can be modeled in, and hence serves as an internal language for, any Grothendieck $(\infty, 1)$ -topos [Shu19]. It builds on Martin-Löf’s dependent type theory by adding Voedvodsky’s univalence axiom and a range of higher inductive types [Uni13]. Martin-Löf’s identity types equip every type with a proof-relevant coherent equivalence relation which is respected by every construction in type theory [Lum09]. HoTT has proven useful as a tool for synthetic homotopy theory and is well suited for formalization using proof assistants.

Proof assistants are well-tuned to support replacing equal elements by equal elements, where equality is reified by the intensional identity type within type theory. Accordingly, if two distinct terms can be identified, they can be swapped out for each other in large proofs without further effort. In HoTT, the identity type becomes far richer and, in particular, elements of the universe become identified whenever they are equivalent. Accordingly, users of proof assistants based on HoTT can swap out e.g., an implementation of the integers well-suited for reasoning with an equivalent version tuned for efficient computation without additional effort. This offers the same convenience to types that function extensionality grants functions. Angiuli et al. [Ang+21], for instance, show that this can be used to internalize some applications of parametricity but, crucially, without eliminating standard models which do not support the full apparatus of parametricity.

A type theory for groupoids makes it far easier to manipulate equality, but what about formalization challenges which are fundamentally asymmetric? For a toy example, consider an algorithm traversing a list to sum its elements $\text{sum} : (A : \text{Monoid}) \rightarrow \text{List } A \rightarrow A$. Univalence and one of its important consequences, the structure identity principle, tell us that sum must respect monoid isomorphisms. But far more is true: sum commutes with all monoid homomorphisms. To prove this we must (1) formulate how a monoid homomorphism $f : A \rightarrow B$ induces a map $\text{List } f : \text{List } A \rightarrow \text{List } B$

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and (2) show that $\text{sum} \circ \text{List } f = f \circ \text{sum}$. Neither task follows from univalence as f need not be invertible and univalence handles only symmetric relations.

1.1 A type theory for categories

The above example would be possible in a version of type theory where types encoded not just groupoids but *categories*: a directed type theory. That is, each type would come equipped with a notion of homomorphism (along with composition, etc.) and each term in the type theory would be bound to automatically respect homomorphisms e.g., be functorial. Aside from the benefits to formalization, it is particularly desirable to find a directed version of HoTT where types would encode ∞ -categories¹ [Joy08; Lur09; Cis19; RV22]; ∞ -category theory is an important area of mathematics but whose foundations are well-known to be cumbersome. It is conjectured that directed homotopy theory could serve as the basis for a more usable and formalizable foundation of this field. Many such theories (both homotopical and not) have been studied over the years [LH11; War13; Nuy15; Nor18; Kav19; Nuy20; WL20; ANvdW23; KS23; NA24; Neu25].

A key obstruction to this program is that (∞ -)categories do not behave well enough to support a model of type theory where every type is a category. For instance, Π -types do not always exist because the category of categories is not locally cartesian closed. Most directed type theories therefore change how type theory works to e.g., allow only certain kinds of Π -types and dependence. We will focus on a different approach introduced by Riehl and Shulman [RS17]: *simplicial type theory* (STT). The key insight is to not require that every type is an ∞ -category, but instead a more flexible object from which we can carve out genuine ∞ -categories using two definable predicates.

STT extends HoTT with a new type to probe the implicit categorical structure each type possesses: the *directed interval* \mathbb{I} . Riehl and Shulman [RS17] further equip it with the structure of a bounded linear order $(\wedge, \vee, 0, 1)$. One can then use \mathbb{I} to access e.g., the morphisms a to b in A by studying ordinary functions within type theory $f : \mathbb{I} \rightarrow A$ such that $f(0) = a$ and $f(1) = b$.

Early evidence [RS17; Rie23; Rie25; Buc19; BW23; Wei24c; Wei24b; Wei24a; Mar24; Mar25] suggests that simplicial type theory approaches the desired usable foundations for ∞ -category theory. A number of definitions and theorems from classical ∞ -category theory have been ported to STT and the proofs are shorter and more conceptual. Even better, Kudasov’s experimental proof assistant Rzk [Kud23] for STT has shown that the arguments for e.g., the Yoneda lemma are simple enough to be formalized and machine-checked [KRW04].

Convention 1.1. For the remainder of this paper, we shall be concerned only with ∞ -categories and constructions upon them. Accordingly, hereafter we largely drop the “ ∞ -” prefix and speak simply of categories, groupoids, etc. except in those few situations where it would cause ambiguity.

Simplicial type theory, a reprise. A brief description of simplicial type theory is in order. As mentioned, every type A in STT has a notion of homomorphism: functions $\mathbb{I} \rightarrow A$. However, in arbitrary types these do not really behave like homomorphisms e.g., they need not compose.

Suppose we are given $f, g : \mathbb{I} \rightarrow A$ such that $f 1 = g 0$. A composite h ought to be a homomorphism such that $h 0 = f 0$ and $h 1 = g 1$, but not every such h satisfying just these conditions ought to be a composite. In particular, further data is required to connect h with f and g . Classically, all of this is encoded by a 2-simplex H (see the left diagram in Fig. 1). Inside simplicial type theory, we represent such 2-simplices as maps $\Delta^2 \rightarrow A$ where $\Delta^2 = \{(i, j) : \mathbb{I} \times \mathbb{I} \mid i \geq j\}$ (c.f., the shaded portion of the right-hand diagram of Fig. 1).

In particular, a 2-simplex $H : \Delta^2 \rightarrow A$ witnesses that $H(-, 0)$ and $H(1, -)$ can be composed to obtain $\lambda i. H(i, i)$. It is convenient to isolate the subtype $\Lambda_1^2 = \{(i, j) \mid i = 1 \vee j = 0\} \subseteq \Delta^2 \subseteq \mathbb{I} \times \mathbb{I}$.

¹Specifically, $(\infty, 1)$ -categories: categories whose morphisms form an ∞ -groupoid.

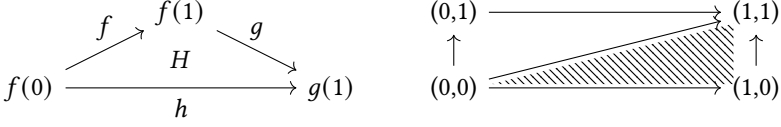


Fig. 1. Illustrations of simplices

Unfolding, a map $\Lambda_1^2 \rightarrow A$ corresponds to a pair of composable arrows f, g . Accordingly, every pair of composable arrows in A admits a unique composite i.e., A is *Segal* if $(\Delta^2 \rightarrow A) \simeq (\Lambda_1^2 \rightarrow A)$.

Segal types already possess enough structure to behave like categories e.g. it follows that composition is associative and unital just from the Segal condition. Unfortunately, they may suffer from an excess of data: they come equipped with two notions of sameness. Namely, $a, a' : A$ may be regarded as the same when $a = a'$ or when there is an invertible homomorphism from a to a' . In keeping with our pursuit of the structure homomorphism principle, we shall be interested in types where these coincide i.e. where $A \rightarrow \sum_{f:A^0} \text{isIso}(f)$ is an equivalence. We say such a type is *Rezk* if it is Segal and satisfies this condition. An important result of Riehl and Shulman [RS17] is that Rezk types adequately model the standard notion of ∞ -category [Rez01]: a theorem proven in STT about a Rezk type describes a valid result for ordinary ∞ -categories.²

Directed univalence. However, simplicial type theory is not a panacea for replacing classical ∞ -category theory. Presently, it is really only suitable for studying “formal” questions and, surprisingly, it is unknown how to construct a non-trivial closed Rezk type within STT. Crucially, STT lacks an equivalent to the category of groupoids (the ∞ -categorical version of the category of sets). Not only does this mean that STT faces severe limitations on what theorems can be *stated*, it is presently impossible to exploit directed path types when formalizing. Returning to our original example with sum, STT would automatically handle (1) and (2) if there was a type of monoids Monoid where directed paths were monoid homomorphisms, but such a definition is presently out of reach.

Our central contribution is to overcome these challenges by extending STT with new reasoning principles and constructing a Rezk type \mathcal{S} whose objects correspond to groupoids (i.e., Rezk types where every homomorphism is invertible) and whose homomorphisms are functions. This last requirement is termed *directed univalence*:

Definition 1.2. A universe \mathcal{S} is *directed univalent* if $\mathbb{1} \rightarrow \mathcal{S}$ is isomorphic to $\sum_{A,B:\mathcal{S}} A \rightarrow B$ over $\mathcal{S} \times \mathcal{S}$.

Before discussing our approach, we survey a few consequences of this result. Once \mathcal{S} is available, a number of applications of STT snap into focus. For instance, one can isolate subcategories of \mathcal{S} such as the category of sets $\mathcal{S}_{\leq 0}$ and the category of propositions $\mathcal{S}_{\leq -1}$. Using the ordinary constructions of type theory, one can parlay these into our aforementioned category of monoids:

$$\text{Monoid} = \sum_{A:\mathcal{S}_{\leq 0}} \sum_{\epsilon:A} \sum_{\cdot:A \times A \rightarrow A} \text{isAssociative}(\cdot) \times \text{isUnit}(\cdot, \epsilon)$$

The only difference in this definition from the standard one seen in ordinary type theory is the replacement of \mathcal{U} by $\mathcal{S}_{\leq 0}$. However, with just this change we are able to prove the following result:

Lemma 7.12. *If $F, G : \text{Monoid} \rightarrow \mathcal{S}$ and $\alpha : (A : \text{Monoid}) \rightarrow F(A) \rightarrow G(A)$ then α is natural i.e. if $f : A \rightarrow B$ is a monoid homomorphism, then $\alpha(B) \circ F(f) = G(f) \circ \alpha(A)$.*

²In fact, combined with general results on HoTT [Shu19; Wei22] they model *internal* ∞ -categories in an ∞ -topos [Mar22; MW23; Cis+24].³

In particular, choosing $F = \text{List}$, $G = \text{id}$, and $\alpha = \text{sum}$ yields our desired earlier example.

Replacing Monoid with Ring, one could derive a similar theorem to argue that given a numerical algorithm $f : (R : \text{Ring}) \rightarrow R^n \rightarrow R$ then the parity of its output (when applied to \mathbb{Z}) depends on the parity of its inputs, as f commutes with the map $\mathbb{Z} \rightarrow \mathbb{Z}/2$. These are instances of a *directed* version of the structure identity principle, the structure homomorphism principle (SHP) [CD13; Uni13; Ahr+22; WL20]: if C is a type of algebraic structures, its homomorphisms coincide with classical morphisms of those structures. Consequently every term and type using C is therefore automatically bound to be functorial and respect these classical morphisms. It was observed by e.g., Weaver and Licata [WL20] that SHP could be used to ease formalization efforts and we provide the first complete examples of this and by proving SHP occurs for a wide class of structures.

More broadly, just as HoTT allowed us to internalize parametricity results based on equivalence relations, STT allows us to internalize parametricity arguments based on naturality. From this, we can also recover a classic result:

Lemma 7.2. *If $f : (A : \mathcal{S}) \rightarrow A \rightarrow A$ then $f = \lambda a. a$.*

We may summarize these results by the slogan “ \mathcal{S} is a type which must be used *covariantly*.” In particular, any type depending on \mathcal{S} (or types derived from it) must be functorial in this argument.

Recreating parametricity arguments, however, is far from the only use of \mathcal{S} . Just as we defined Monoid, we can define various categories critical for ∞ -category theory, such as the category of partial orders, the simplex category, the category of finite sets, etc. Using these, we present the first steps towards formalizing *higher algebra* (one of the main applications of ∞ -category theory) within type theory. Higher algebra is most often encountered by type theorists in the form of the *coherence problem* and, from this point of view, using \mathcal{S} we are able to give definitions of infinitely coherent monoids, groups, etc. Fundamentally, having just \mathcal{S} available throws open the door to defining a wide variety of derived categories and all the applications this entails.

1.2 Constructing \mathcal{S}

In a certain sense, the difficulty with \mathcal{S} to STT is not so much in its addition—we could always postulate a type $X : \mathcal{U}$ along with terms for the Segal and Rezk axioms, declare it to be \mathcal{S} , and call it a day! The challenge comes in finding a complete API for \mathcal{S} within STT that, when established, allows us to prove all expected results and determines \mathcal{S} up to a contractible choice of isomorphism. This is where ∞ -categories prove substantially more complex than 1-categories. It no longer suffices to specify objects and morphisms to define \mathcal{S} , we must also specify the higher simplices needed for coherent composition. Thus, even if we set aside the distasteful nature of simply adding axioms to construct \mathcal{S} , we would be left with the task of adding an *infinite* number of axioms on top of e.g., directed univalence to fully specify its behavior. This is a famous problem of ∞ -category theory where nearly all constructions must be carried out indirectly through heavy machinery.

Our main theorem therefore is to construct \mathcal{S} internally and thereby provide a complete API for its use. We do this by adapting the methods of Licata, Shulman, and Riley [LSR17] and Weaver and Licata [WL20] to prove one of the most widely-used results in ∞ -category theory, the straightening–unstraightening equivalence [Lur09; HM16; Cis19; CN22], inside of type theory. Roughly, we define \mathcal{S} and prove that the type $X \rightarrow \mathcal{S}$ is equivalent to the subtype of $X \rightarrow \mathcal{U}$ spanned by *amazingly covariant families*. That is, a map $X \rightarrow \mathcal{S}$ corresponds to a type family over X which is covariant in X as well as the context i.e., *amazingly covariant* [Ril24].

We show that all the central properties of \mathcal{S} follow from this description. For instance, we are able to show that \mathcal{S} is closed under the expected operations (limits, colimits, dependent sums, and certain dependent products) and, most importantly, we prove the directed univalence axiom.

1.3 Extending simplicial type theory to triangulated type theory

The central challenge is giving an adequate definition of *amazingly covariant* families: types $\Gamma \vdash A : X \rightarrow \mathcal{U}$ which are covariant not only in X , but the entire context Γ . This second condition, however, cannot be expressed inside of simplicial type theory. Similar situations have arisen in many contexts within HoTT [Sch13; SS12; Shu18; MR23] and, as in prior work, we address this lack of expressivity by extending simplicial type theory by a collection of *modalities* to capture amazing covariance.

In fact, even without amazing covariance we are led to modal simplicial type theory or indeed, modal versions of any type theory seeking to internalize directed univalence. The reason why can be summed up in a single word: contravariance. It is all well and good to have a type whose use is automatically covariant, but common operations on the universe (e.g., $X \mapsto X \rightarrow \text{Bool}$) are simply not covariant, and some (e.g. $X \mapsto X \rightarrow X$) are neither co- nor contravariant. As it stands, \mathcal{S} can only be used covariantly and so we cannot express these important and natural operations. To rectify this, we extend STT with modalities which allow us to express *contravariant* dependence on \mathcal{S} as well as *invariant* dependence. Both of these modalities have central positions within synthetic category theory: the first sends a category to its opposite and the second sends a category to its underlying groupoid of objects. While neither operation can be realized as a function $\mathcal{U} \rightarrow \mathcal{U}$ [Shu18], both of these operations can be included as modalities [Gra+21].

Having accepted that some modalities are necessary for simplicial type theory, it is then natural to ask what other modalities must be added in order to internally define amazing covariance and \mathcal{S} . Following Licata et al. [Lic+18], we would like to include a modality which behaves like the right adjoint to $A \mapsto (\mathbb{1} \rightarrow A)$; the so-called amazing right adjoint to $\mathbb{1} \rightarrow -$. In op. cit., the intended model (cubical sets) had such a modality but in the standard model of simplicial type theory, no such right adjoint exists. Accordingly, we could add such a modality to simplicial type theory, but we would have no means by which to justify it. In order to address this, we must also weaken the standard model of simplicial type theory and, with it, the assumed structure on $\mathbb{1}$. Rather than postulating a totally ordered $\mathbb{1}$, we only ask that $\mathbb{1}$ be a bounded distributive lattice where $0 \neq 1$. Semantically, this corresponds to shifting from simplicial spaces—the standard model—to the larger category of cubical⁴ spaces. Within this new category, the necessary right adjoint exists and we can justify the addition of the necessary modality. In order to manipulate these new modalities and relaxed interval, we also axiomatize several general facts from the cubical spaces model. All told, we work within a version of MTT [Gra+21] (to account for modalities) and with a less structured interval $\mathbb{1}$. We term the result *triangulated type theory* TT_{\square} .

Within TT_{\square} , we isolate *simplicial types*, those which *believe* the interval to be totally ordered. Simplicial types “embed” STT into TT_{\square} and we are eventually interested only in these types (in fact, mostly in simplicial Rezk types). However, the presence of non-simplicial types is crucial to allow for the constructions needed to define \mathcal{S} —even though \mathcal{S} will itself turn out to be simplicial Rezk.

Finally, we note that while MTT enjoys canonicity [Gra22], adding axioms (univalence, $\mathbb{1}$, etc.) obstructs computation and so canonicity does not hold for TT_{\square} . Accordingly, TT_{\square} is closer to “book HoTT” [Uni13] than cubical type theory [Coh+17]. We leave it to future work to develop computational versions of our new axioms and integrate existing computational accounts of univalence in MTT [Aag+22].

⁴Technically, we work within the category of Dedekind cubical spaces. See Section 4.

1.4 Contributions

We contribute $\mathbb{T}\mathbb{T}_{\square}$, a modal extension of simplicial type theory, and use it to construct a directed univalent universe of groupoids \mathcal{S} . In so doing, we construct the first non-trivial examples of categories within simplicial type theory. More specifically:

- We identify several general and reusable reasoning principles with which to extend STT.
- We prove that \mathcal{S} satisfies (directed) univalence, as well as the Segal and Rezk conditions.
- We construct *full subcategories* purely internally and isolate important subcategories of \mathcal{S} .
- We build numerous important classical examples of categories e.g., presheaves, spectra, partial orders, and other (higher) algebraic categories from \mathcal{S} .

Finally, we crystallize a conjectured *structure homomorphism principle* which can be used to recover various parametricity arguments as well as automatically discharge functoriality goals and proof obligations. We give the first complete example applications of this principle.

We have endeavored throughout this paper to make most proofs reasonably explicit. This is not only for the sake of rigor, but because a major contribution of our synthetic approach with both STT and $\mathbb{T}\mathbb{T}_{\square}$ is the comparative simplicity of the proofs. Crucially, no knowledge of ∞ -categories or the semantics of homotopy type theory is required by our key arguments. Even the most complex arguments in [Section 6](#) take up only half of page and are possible to follow to those experienced with (modal) type theory. Ideally, we would substantiate this claim by formalizing our arguments in a proof assistant, but there is presently no suitably general implementation of modal type theory.

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2 A primer on simplicial and modal type theory

Before diving into the construction of the universe of groupoids, we recall some of the details of simplicial type theory from Riehl and Shulman [\[RS17\]](#) and its modal extension. Both simplicial type theory and the modal type theory we combine it with are extensions of homotopy type theory and so, while we assume some familiarity with HoTT, we recall some of the basic notions “book HoTT” as described by the Univalent Foundations Program [\[Uni13\]](#) to fix our chosen notation.

Recall that book HoTT extends an ordinary type theory with the univalence axiom. For us, this ordinary will be intensional Martin-Löf type theory with a hierarchy of universes $\mathcal{U}_0 : \mathcal{U}_1 : \dots$ etc. We will further assume that these universes are cumulative and closed under all relevant connectives. Notably, we assume our universes are closed under propositional truncation and—in

one instance—pushouts. We do not require that these higher inductive types satisfy any particular definitional equalities, as this is presently not supported by the interpretation of HoTT into an ∞ -topos [LS20; Shu19].

We will follow Univalent Foundations Program [Uni13] and write $a =_A b$ (or, even more tersely) $a = b$ for the intensional identity type. Moreover, if $p : a = b$ we shall write $p_* : B(a) \rightarrow B(b)$ for the transport function defined by path induction on p associated with $B : A \rightarrow \mathcal{U}$ and, on occasion, $p \bullet q$ for the concatenation of two paths.

Most importantly, we shall assume that each universe \mathcal{U}_i satisfies Voevodsky’s univalence axiom. In particular, if we write $A \simeq B$ for the subtype of $A \rightarrow B$ spanned by equivalences, we assume that the following canonical map is an equivalence:

$$\text{ua} : (A B : \mathcal{U}_i) \rightarrow (A = B) \rightarrow (A \simeq B)$$

We refer the reader again to Univalent Foundations Program [Uni13] for a thorough discussion of this axiom. Finally, we recall a few crucial notations from op. cit. which we shall repeatedly use:

$$\begin{aligned} \text{isContr}, \text{isHProp}, \text{isHSet} : \mathcal{U} \rightarrow \mathcal{U}; \quad \text{isContr } A &= \sum_{a:A} \prod_{b:B} a = b, \\ \text{isHProp } A &= \prod_{a,b:A} \text{isContr}(a = b), \quad \text{isHSet } A = \prod_{a,b:A} \text{isHProp}(a = b) \end{aligned}$$

These predicates respectively isolate (1) types which behave like Unit , i.e. are contractible, (2) types which behave like propositions, and (3) types which behave like discrete spaces (i.e. sets). In homotopical parlance, these are the (-2) -, (-1) -, and 0 -truncated types. In fact, we can define each of these predicates as instances of a more general $\text{hasHLevel} : \text{Nat} \rightarrow \mathcal{U} \rightarrow \text{HProp}$, but we do not have need for this additional generality. Each of these induce subtypes of the universe e.g., $\text{HProp} = \sum_{A:\mathcal{U}} \text{isHProp } A$. For instance, we may speak of families of propositions over A (predicates) using maps $A \rightarrow \text{HProp}$.

2.1 Simplicial type theory and basic category theory

We now turn to simplicial type theory, an extension of HoTT designed to reason about simplicial spaces and, through them, ∞ -categories. The main axiom of simplicial type theory asserts the existence of a type which internalizes the representable Δ^1 or, equivalently, the category with two objects and one non-trivial morphism connecting them:

Definition 2.1. Core simplicial type theory STT extends homotopy type theory with the following:

- (1) A *directed interval* type $\mathbb{I} : \text{HSet}$
- (2) The operations and equations shaping \mathbb{I} into a bounded total order $(0, 1, \leq)$.

The precise form of this axiom is subject to numerous variations. For instance, in the original incarnation of simplicial type theory [RS17] featured a bespoke judgmental structure for \mathbb{I} which enabled additional definitional equalities. In order to keep the system simple in anticipation of adding various modalities to it presently, we have eschewed this structure so that \mathbb{I} is an ordinary type. Moreover, since we shall shortly be interested in a model of simplicial type theory in *cubical* spaces, we have only required that \mathbb{I} be a bounded distributive lattice rather than a linear order.

Using the lattice structure on \mathbb{I} , we can now specify the common simplicial shapes used to model composition in ∞ -category theory e.g. Δ^n :

$$\Delta^0 := \text{Unit} \quad \Delta^{n+1} := \{(i_1, \dots, i_{n+1}) : \mathbb{I}^{n+1} \mid i_1 \geq i_2 \geq \dots \geq i_{n+1}\} \quad \Lambda_1^2 := \{(i, j) : \mathbb{I}^2 \mid i = 1 \vee j = 0\}$$

One can also give general descriptions of the boundaries $\partial\Delta^n$ and the (n, k) -horns Λ_k^n , for $n \geq 0$ and $0 \leq k \leq n$ [RS17, Section 3]. We use these to define categories and related structures.

Definition 2.2. Given $a, b : A$, the type of *homomorphisms* or *arrows* from a to b is given by

$$\text{hom}_A(a, b) := \sum_{f : \mathbb{1} \rightarrow A} f \cdot 0 = a \times f \cdot 1 = b$$

In other words, $\text{hom}_A(a, b)$ is the (homotopy) fiber of $A^{\mathbb{1}} \rightarrow A \times A$ over (a, b) . For convenience, we suppress the forgetful map $\text{hom}_A(a, b) \rightarrow (\mathbb{1} \rightarrow A)$ and so will write $f(i)$ when $f : \text{hom}_A(a, b)$ or similar.

Notation 2.3. We write ε for the tautological homomorphism $\text{hom}(0, 1)$ in $\mathbb{1}$ induced by id .

We can relativize the notion of homomorphisms to dependent types:

Definition 2.4. Given $a, b : A$ and $f : \text{hom}_A(a, b)$, for a type family $P : A \rightarrow \mathcal{U}$, a *dependent homomorphism* from $x : P a$ and $y : P b$ over f is given by

$$\text{hom}_P^f(x, y) := \sum_{\varphi : (i : \mathbb{1}) \rightarrow P(f \cdot i)} ((\pi_2 f)_*(\varphi \cdot 0) = x) \times ((\pi_3 f)_*(\varphi \cdot 1) = y)$$

Note that we must transport by the identifications $\pi_2 f : f(0) = a$ and $\pi_3 f : f(1) = b$ in order to ensure that these equalities are type-correct.⁵

With the apparatus of morphisms to hand, we may recall the following definition of *pre-categories* i.e. *Segal types* from the introduction:

Definition 2.5. $A : \mathcal{U}$ is *Segal* if the canonical map $i : (\Delta^2 \rightarrow A) \rightarrow (\Delta_1^2 \rightarrow A)$ is an equivalence.

Notation 2.6. If A is Segal and $f : \text{hom}(a, b)$, $g : \text{hom}(b, c)$, we write $g \circ f$ for the map $\mathbb{1} \rightarrow A$ given by $\lambda t. (i^{-1}(f, g))(t, t)$ i.e. the long edge of the triangle obtained by extending $(f, g) : \Delta_1^2 \rightarrow A$ to $\Delta^2 \rightarrow A$. This operation is automatically associative and constant functions $\mathbb{1} \rightarrow A$ (identity homomorphisms) are units for \circ . A major benefit of working in simplicial type theory is that such a composition does exist, even though composites are defined only up to a contractible choice.

Definition 2.7. We say an arrow $f : \text{hom}(a, b)$ in a Segal type A is an *isomorphism* if the following type is inhabited: $\text{isIso}(f) := \sum_{g, h : \text{hom}(b, a)} (g \circ f = \text{id}_a) \times (f \circ h = \text{id}_b)$.

Note that $\text{isIso}(f)$ is a proposition and we denote the induced subtype of $\text{hom}(a, b)$ by $a \cong_A b$. With the definition of isomorphism to hand, we can properly define *categories* and *groupoids*:

Definition 2.8. If A is Segal, we say A is a *category/Rezk-complete* if the following map (defined by path induction) is an equivalence: $\text{IdToIso} : (a, b : A) \rightarrow (a = b) \rightarrow (a \cong b)$

Remark 2.9. We note that the proposition $\text{isIso}(f)$ requires that f have a section and a retraction; a common definition of equivalence in HoTT. We shall return to this point in [Section 6](#) when we prove that our directed-univalent universe satisfies the Rezk condition.

Definition 2.10. A type A is *groupoid* or a *space* or $\mathbb{1}$ -*null* if $(a = b) \rightarrow (a \rightarrow b)$ is an equivalence.⁶

Lemma 2.11. *A type is a groupoid if and only if it is a category where every arrow is an isomorphism.*

Intuitively, a type is Rezk when it satisfies a kind of univalence condition: isomorphism is identity. In the intended model of TT_{\square} , they correspond to complete Segal spaces, in turn, model $(\infty, 1)$ -categories. Op. cit. further show that maps between Segal types are automatically *functors* i.e. they preserve composition and identities.

⁵These transports are precisely what the judgmental extensions of Riehl and Shulman [RS17] aim to avoid. Fortunately they do not occur so frequently as to be a major impediment in this paper.

⁶The terminology “ $\mathbb{1}$ -null” stems from Rijke, Shulman, and Spitters [RSS20]; it is equivalent to requiring that the constant map $A \rightarrow (\mathbb{1} \rightarrow A)$ is an equivalence.

2.2 Multimodal type theory

As mentioned in [Section 1](#), we must extend type theory with various *modalities* in order to define \mathcal{S} . We shall do this by “rebasin” simplicial type theory atop MTT [[Gra+21](#)], a general framework for modal type theory. In particular, we shall take our base type theory to not just be intensional Martin-Löf type theory, but a slightly richer theory which includes various modalities alongside the ordinary constructors of dependent type theory. Since MTT is already a complex type theory without any of the additional axioms of homotopy or simplicial type theory, we give a brief overview of the theory now. We refer the reader to Gratzter et al. [[Gra+21](#)] or Gratzter [[Gra23](#), Chapter 6] for a more thorough introduction to the theory. We will also explain MTT as it is used in this paper: in the same informal style that we will use type theory generally. Accordingly, we will not focus overmuch on the substitution calculus of MTT or other aspects of its metatheory which need not concern us presently. An exception to this general pattern is the description of the semantics of $\mathbb{T}\mathbb{T}_{\square}$, but we will briefly recall the model theory of MTT at that point.

We must immediately note that MTT is not, properly speaking, a type theory. Rather, MTT is a *framework* for modal type theories: a user picks a mode theory—a 2-category describing their modalities—and MTT produces a type theory for working with this collection of modalities. For this exposition, let us fix \mathcal{M} an arbitrary 2-category where we think of objects (*modes*) m, n as type theories connected by the 1-cells (*modalities*) μ, ν . The 2-cells α, β encode transformations between modalities enabling us to control e.g., whether μ is a comonad. In fact, for our particular use case we shall have at most one 2-cell between any pair of modalities and exactly one mode. For simplicity, we shall assume the same to be true of \mathcal{M} and we write $\mu \leq \nu$ when there exists a (necessarily unique) 2-cell from μ to ν . Instantiating MTT with \mathcal{M} yields a type theory which includes a modal type for each μ in the mode theory and, furthermore, these modal types are *2-functorial*.

The basic modification MTT makes to the type theory is to change the form of variables in the context. A context is no longer simply a telescope of bindings $x : A$. Instead, each declaration is annotated by a pair of modalities $x :_{\mu/\nu} A$. The annotation μ/ν signifies that x was constructed under the μ modality and, presently, we are working to construct an element of the ν modality.

Notation 2.12. In a declaration $x :_{\mu/\nu} A$ we shall often omit μ or ν if they are the identity e.g., $x :_{\mu} A$ or $y : B$ rather than $x :_{\mu/\text{id}} A$ or $y :_{\text{id}/\text{id}} B$.

Both halves of the annotation μ/ν restrict how variables are used to prevent terms from illegally escaping or entering modalities and, roughly, we are allowed to use a variable when they cancel.

$$\frac{\mu \leq \nu \quad x :_{\mu/\nu} A \in \Gamma}{\Gamma \vdash x : A} \qquad \frac{\Gamma/\mu \vdash a : A \quad \Gamma, x :_{\mu/\text{id}} A \vdash b(x) : B(x)}{\Gamma \vdash b[a/x] : B[b/x]}$$

In the above, Γ/μ denotes the context with the same variables as Γ but where $x :_{\nu/\nu_0} A$ is replaced by $x :_{\nu/(\nu_0 \circ \mu)} A$. Note we have presented only the relevant and simpler substitution rule allowing us to discharge an assumption with the “denominator” of an annotation is the identity. Note also that MTT does not alter the actual definition of substitution from type theory—one merely proves after the fact that ordinary substitution does not result in terms using inaccessible variables.

These annotations are also used to introduce the modal types associated with each μ . For instance, suppose we have a modality μ , and we intend to form the modal type $\langle \mu \mid A \rangle$. This is well-formed in context Γ just when A is well-formed in the context Γ/μ . Similarly, we can form an element of the modal type $\text{mod}_{\mu}(a) : \langle \mu \mid A \rangle$ in context Γ just when $a : A$ in the context Γ/μ :

$$\frac{\Gamma/\mu \vdash A}{\Gamma \vdash \langle \mu \mid A \rangle} \qquad \frac{\Gamma/\mu \vdash a : A}{\Gamma \vdash \text{mod}_{\mu}(a) : \langle \mu \mid A \rangle}$$

The elimination rule for $\langle \mu \mid - \rangle$ papers over the difference between $a :_{\nu \circ \mu / \text{id}} A$ and $a' :_{\nu / \text{id}} \langle \mu \mid A \rangle$:

$$\frac{\Gamma / \nu \circ \mu \vdash A \quad \Gamma, y :_{\nu / \text{id}} \langle \mu \mid A \rangle \vdash B(y) \quad \Gamma, x :_{\nu \circ \mu / \text{id}} A \vdash b(x) : B[\text{mod}_{\mu}(x)/y] \quad \Gamma / \nu \vdash a : \langle \mu \mid A \rangle}{\Gamma \vdash \text{let}_{\nu} \text{mod}_{\mu}(x) \leftarrow a \text{ in } b(x) : B[a/y]}$$

$$\text{let}_{\nu} \text{mod}_{\mu}(x) \leftarrow \text{mod}_{\mu}(a_0) \text{ in } b(x) = b[a_0/x]$$

In particular, if we are attempting to construct a term using a variable x of type $\langle \mu \mid A \rangle$, we may reduce to the case where $x = \text{mod}_{\mu}(x_0)$ for a fresh variable $x_0 :_{\mu} A$. More technically, this amounts to a certain canonical map being weakly orthogonal to all types. We shall revisit this perspective in the discussion of semantics.

Already, these rules are sufficient to prove the following facts that we shall use ubiquitously:

Lemma 2.13. *If $\Gamma / \nu \circ \mu \vdash A$ then $\langle \nu \mid \langle \mu \mid A \rangle \rangle \simeq \langle \nu \circ \mu \mid A \rangle$ and if $\Gamma \vdash B$ then $\langle \text{id} \mid B \rangle \simeq B$.*

We have already seen that $\mu \leq \nu$ allows us to access variables under μ/ν . This, combined with the elimination rule for modal types, allows us to produce a function introduces a function $\langle \mu \mid - \rangle \rightarrow \langle \nu \mid - \rangle$. In order to make this well-formed, we note the following admissible principle in MTT, akin to the admissibility of weakening in ordinary type theory:

Lemma 2.14. *If $\Gamma / \nu \vdash a : A$ and $\mu \leq \nu$ then $\Gamma / \mu \vdash a : A$.*

Lemma 2.15. *If $\alpha : \mu \rightarrow \nu$ and $\Gamma / \mu \vdash A$ then there is a map $\text{coe}^{\mu \leq \nu} : \langle \mu \mid A \rangle \rightarrow \langle \nu \mid A \rangle$.*

PROOF. Though this is an elementary result, we give a proof to highlight the process of working in MTT. Suppose we are given $x : \langle \mu \mid A \rangle$, we must construct a term of type $\langle \nu \mid A \rangle$. Using the elimination principle for $\langle \mu \mid A \rangle$, we may fix $x_0 :_{\mu} A$ and assume that $x = \text{mod}_{\mu}(x_0)$. Next, using the introduction rule for $\langle \nu \mid A \rangle$, it suffices to construct an element of A , though we must update the annotations on x and x_0 to $x_0 :_{\mu/\nu} A$ and $x :_{\text{id}, \nu} \langle \mu \mid A \rangle$. Finally, we may use the variable rule to access $x_0 : A$ as required. All told then, the full term is given as follows:

$$\text{coe}^{\mu \leq \nu} = \lambda x. \text{let } \text{mod}_{\mu}(x_0) \leftarrow x \text{ in } \text{mod}_{\nu}(x_0)$$

This pattern of binding a variable of type $\langle \mu \mid A \rangle$ and immediately pattern-matching upon it is exceptionally common. Accordingly, we adopt the following “pattern-matching” style notation for convenience:

$$\text{coe}^{\mu \leq \nu}(\text{mod}_{\mu}(x_0)) = \text{mod}_{\nu}(x_0) \quad \square$$

Notation 2.16. We shall also have occasion to use the convenience feature of modalized dependent products $(a :_{\mu} A) \rightarrow B(a)$ which abstract over $a :_{\mu} A$ directly rather than $a :_{\text{id}} \langle \mu \mid A \rangle$ to allow us to avoid immediately pattern-matching on a . In particular, $(a :_{\mu} A) \rightarrow B(a)$ is equivalent to $(a :_{\text{id}} \langle \mu \mid A \rangle) \rightarrow (\text{let } \text{mod}_{\mu}(a_0) \leftarrow a \text{ in } B(a_0))$. We shall most often use this when working informally to type theory. For instance, if we write “given $a :_{\mu} A$, there exists an element of $B(a)$ ” this should be interpreted as denoting the type $(a :_{\mu} A) \rightarrow B(a)$.

3 Triangulated type theory

As already mentioned, part of our strategy is to replicate the argument of Weaver and Licata [WL20] in STT combined with MTT and build a directed univalent universe of groupoids. However, there is a fundamental problem with this approach: Weaver and Licata [WL20] rely on a particular modality (the right adjoint to $\mathbb{1} \rightarrow -$) in their construction, and the standard model of simplicial type theory in $\text{PSh}(\Delta)$ simply does not have an analog to this modality. Thus, we need more than a combination of STT and MTT, we need a new system which admits a model where $\mathbb{1} \rightarrow -$ is a left adjoint.

To motivate our solution, let us further consider the intended model of STT in simplicial spaces for a moment. Up to the complexity needed to model homotopy type theory, these are simplicial sets i.e., presheaves on Δ , the category of finite, inhabited total linear orders. The interval \mathbb{I} is realized by $\mathbf{y}([1])$. Our problem then amounts to the fact that exponentiation by $\mathbf{y}([1])$ does not have a right adjoint.⁷ However, there is a category closely related to Δ which has also received a great deal of attention by type theorists interested in cubical type theory: the category of (Dedekind) cubes \square , the full subcategory of partial orders spanned by $\{0 \leq 1\}^n$ for all n . The category \square enjoys two properties which make it interesting for simplicial type theorists: there is a fully faithful functor $\text{PSh}(\Delta) \rightarrow \text{PSh}(\square)$ which preserves the interval and admits both left and right adjoints [Sat19; KV20; SW21], and within $\text{PSh}(\square)$ exponentiating by the interval is a left adjoint.

Accordingly, we introduce a relaxation of simplicial type theory intended to capture (the homotopical version of) $\text{PSh}(\square)$. Within this type theory, we can recover simplicial type theory by studying those types which are in the image of the aforementioned embedding alongside the amazing right adjoint necessary for constructing our sought-after universe [Lic+18].

Concretely, we work within a version of MTT instantiated with several modalities, further extended by a bounded distributive lattice $\mathbb{I} : \text{HSet}$ which serves as our weakened version of the interval, and a handful of axioms. Notably, we no longer assume that \mathbb{I} is totally ordered and instead ask for it to be a bounded distributive lattice; this is our central deviation from simplicial type theory.

In this section, we introduce *triangulated type theory* as an extension of MTT. For precision, we give a complete listing of the axioms we require (including univalence, the existence of an interval, etc.). Finally, we explore a few elementary consequences of this axiom and produce our first non-trivial examples of categories.

3.1 The definition of triangulated type theory

We begin by describing the particular instantiation of MTT needed for TT_{\square} .

3.1.1 The mode theory. As mentioned in Section 2.2, MTT must be instantiated by a mode theory. In our case, we shall require only one mode m which we shall think of as cubical spaces $\text{PSh}_{\text{Set}}(\square)$. We shall then add the following modalities

- A pair of modalities \flat, \sharp internalizing the global sections comonad and its right adjoint.
- A modality op internalizing the “opposite.”

Intuitively, $\langle \text{op} \mid X \rangle$ is a type with the same points as X , but with all the higher cubes reversed; if there was a line joining x_0 to x_1 in X , then $\langle \text{op} \mid X \rangle$ will have a line joining x_1 to x_0 instead.

On the other hand, $\langle \flat \mid - \rangle$ deletes all (higher) cells from a type, leaving only the underlying groupoid of points (its *groupoid core*). We shall use this modality to define the core of a category and, more generally, use it to isolate discrete categories. Owing to this second point, prior work has often referred to \flat as the *discrete* modality [Shu18; MR23]. Its right adjoint, $\langle \sharp \mid - \rangle$, is slightly less intuitive. Operationally, it deletes all higher cells from a type and then adds in a unique (higher) cell between every collection of points. We caution the reader that while $\langle \flat \mid X \rangle$ is always a groupoid in the sense of Section 2, it is often the case that $\langle \sharp \mid X \rangle$ is not a category even if X was originally a category. This is not unexpected: $\langle \flat \mid - \rangle$ models the core functor sending a category to its underlying groupoid, but this operation when restricted to categories has no right adjoint. It is only in a bigger category such as cubical or simplicial spaces that $\langle \sharp \mid - \rangle$ exists.

⁷This is easiest to check by observing that it does not commute with pushouts.

We require a number of equations and inequalities to force these modalities to behave as expected. In particular, we require the following 2-cells and equations on modalities:

$$\begin{aligned} b \circ b &= b \circ \# = b \circ \text{op} = \text{op} \circ b = b & \# \circ \# &= \# \circ b = \# \circ \text{op} = \text{op} \circ \# = \# \\ \text{op} \circ \text{op} &= \text{id} & b &\leq \text{id} & \text{id} &\leq \# \end{aligned}$$

We refer to the 2-category theory generated by these constraints as \mathcal{M}_{\boxtimes} .

3.1.2 The interval. As mentioned previously, we require an interval in order to capture the simplicial (or, in our case, cubical) structure.

Axiom 1 (The interval). *There exists a bounded distributive lattice $(\mathbb{I} : \text{HSet}, \wedge, \vee, 0, 1)$.*

Our next axiom controls the behavior of the opposite modality on \mathbb{I} :

Axiom 2 (Opposite of \mathbb{I}). *There is an equivalence $\neg : \langle \text{op} \mid \mathbb{I} \rangle \rightarrow \mathbb{I}$ which swaps 0 for 1 and \vee for \wedge .*

Notation 3.1. In various places, it will be convenient to treat \neg as a function $(\text{op} \mid \mathbb{I}) \rightarrow \mathbb{I}$ to avoid spuriously introducing $\text{mod}_{\text{op}}(-)$. The two types $(\text{op} \mid \mathbb{I}) \rightarrow \mathbb{I}$ and $\langle \text{op} \mid \mathbb{I} \rangle \rightarrow \mathbb{I}$ are canonically equivalent and so this causes no ambiguity, see [Notation 2.16](#).

With \mathbb{I} , we are now able to postulate an amazing right adjoint operation to $\mathbb{I} \rightarrow -$. As was noted in Licata et al. [[Lic+18](#)], this operation cannot be defined as a map $\mathcal{U} \rightarrow \mathcal{U}$. Accordingly, we restrict its action to global elements using b .

Axiom 3 (\mathbb{I} is tiny). *The following proposition holds:*

$$(A :_b \mathcal{U}) \rightarrow \sum_{A_{\boxtimes} :_b \mathcal{U}} \sum_{\varepsilon :_b (A_{\boxtimes})^{\boxtimes} \rightarrow A} \prod_{B :_b \mathcal{U}} \text{isEquiv}(\langle b \mid B \rightarrow A_{\boxtimes} \rangle \rightarrow \langle b \mid B^{\boxtimes} \rightarrow A \rangle)$$

In other words, we require that for each element $A :_b \mathcal{U}$, there exists a type A_{\boxtimes} which represents the functor $(-)^{\boxtimes} \rightarrow A$ i.e., a right adjoint to $\mathbb{I} \rightarrow -$.

3.1.3 The simplicial monad. Before moving on to the list of additional axioms that form TT_{\boxtimes} , we must take a moment to discuss an additional construct: the simplicial monad. As motivation, while we have already noted that the interval is not totally ordered, there is a large number of types which “act as though it is.” The simplicial monad isolates and classifies these types.

More precisely, a type is *simplicial* if it satisfies the following predicate:

$$\text{isSimp}(A) = (i j : \mathbb{I}) \rightarrow \text{isEquiv}(\lambda a z. a : A \rightarrow (i \leq j \vee j \leq i \rightarrow A))$$

If a type A satisfies isSimp , this acts as a license to totally order elements of the interval whenever we are constructing an element of A . Furthermore, as the name suggests, simplicial types are those which come from simplicial rather than cubical sets (see [Section 4](#)).

Proposition 3.2 (Rijke, Shulman, and Spitters [[RSS20](#)]). *There is a monad $(\boxtimes : \mathcal{U} \rightarrow \mathcal{U}, \eta, \mu)$ such that:*

- For every $A : \mathcal{U}$, $\text{isSimp}(\boxtimes A)$ holds.
- If B is simplicial, then $\eta^* : (\boxtimes A \rightarrow B) \rightarrow (A \rightarrow B)$ is an equivalence.
- \boxtimes commutes with dependent sums and the identity type.

We refer to \boxtimes as the *simplicial monad*⁸ and write \mathcal{U}_{\boxtimes} for the subtype $\sum_{A : \mathcal{U}} \text{isSimp}(A)$.

Convention 3.3. We reserve the words “category” and “groupoid” for types which are simplicial in addition to satisfying the Segal/Rezk conditions from STT . Accordingly, e.g. *category* signifies a type which is simplicial, Segal, and Rezk complete.

⁸The notation \boxtimes is chosen deliberately: simplicial types are those which believe the square $\mathbb{I} \times \mathbb{I}$ (along with all hypercubes) comes from gluing together a pair of triangles $\Delta^2 \sqcup_{\mathbb{I}} \Delta^2$.

3.1.4 Additional axioms. Finally, we require a handful of additional axioms which either improve the behavior of modalities generally or form a more tight correspondence between our system and our intended model. We offer some intuition for each axiom and note that each is validated by the intended model described in [Section 4](#).

Our first two axioms are general and common assumptions in univalent modal type theory. First, we record the univalence axiom here as previously discussed in [Section 2](#):

Axiom 4 (Univalence). *We assume that each universe \mathcal{U}_i is univalent.*

Next, we assume that each modality $\langle \mu \mid - \rangle$ commutes with identity types.

Axiom 5 (Crisp induction). *For every μ , the canonical map $\text{mod}_\mu(a) = \text{mod}_\mu(b) \rightarrow \langle \mu \mid a = b \rangle$ is an equivalence.*

Remark 3.4. It is open whether [Axiom 4](#) implies [Axiom 5](#). However, in all sensible models, [Axiom 5](#) does hold and its failure to do so is more indicative of the poor behavior of the intensional identity type than anything else.

After these fairly general reasoning principles, we now have a sequence of more simplicial-specific axioms. The first of these links the global sections modality to the interval. In particular, it states that the global sections of a type always form a groupoid.

Axiom 6 ($\mathbb{1}$ detects discreteness). *If $A \vDash_{\mathbb{1}} \mathcal{U}$ then $\langle \mathbb{b} \mid A \rangle \rightarrow A$ is an equivalence (A is discrete) if and only if $A \rightarrow (\mathbb{1} \rightarrow A)$ is an equivalence (A is $\mathbb{1}$ -null).*

The next axiom states that the global points of $\mathbb{1}$ itself are just 0 and 1 and that $0 \neq 1$:

Axiom 7 (Global points of $\mathbb{1}$). *The canonical map $\text{Bool} \rightarrow \mathbb{1}$ is injective and $\text{Bool} \simeq \langle \mathbb{b} \mid \mathbb{1} \rangle$.*

In our intended model, various properties can be proven by “testing” them at the representable presheaves $\mathbf{y}(\{0 \leq 1\}^n)$. We include a version of this idea as an axiom in our theory. Namely, we assert that maps between global types can be tested for invertibility at $\mathbb{1}^n$:

Axiom 8 (Cubes separate). *A map $f \vDash_{\mathbb{1}} A \rightarrow B$ is an equivalence if and only if the following holds:*

$$(n \vDash_{\mathbb{1}} \text{Nat}) \rightarrow \text{isEquiv}(f_* : \langle \mathbb{b} \mid \mathbb{1}^n \rightarrow A \rangle \rightarrow \langle \mathbb{b} \mid \mathbb{1}^n \rightarrow B \rangle)$$

This follows from another possible axiom, *cubes detect continuity*, following Myers and Riley [[MR23](#)]. Note that if A and B are simplicial, one can derive a version of [Axiom 8](#) which replaces $\mathbb{1}^n$ with Δ^n .

It is relatively easy to characterize maps out of $\square A$ as they are closely related to maps out of A itself. It is much harder, however, to characterize $X \rightarrow \square A$. Our next axiom states that in certain favorable cases these, too, coincide with the corresponding situation for A :

Axiom 9 (Simplicial stability). *If $A \vDash_{\mathbb{1}} \mathcal{U}$ then the following map is an equivalence for all $n \vDash_{\mathbb{1}} \text{Nat}$:*

$$\eta_* : \langle \mathbb{b} \mid \Delta^n \rightarrow A \rangle \rightarrow \langle \mathbb{b} \mid \Delta^n \rightarrow \square A \rangle$$

Finally, while simplicial type theory allows us to prove many interesting facts about maps out of the interval, it is far more difficult to prove properties about $X \rightarrow \mathbb{1}$. In order to balance the scales, we follow Cherubini, Coquand, and Hutzler [[CCH23](#)] and add a duality axiom [[Koc14](#); [Ble23](#)] characterizing these maps in certain special cases. Prior to stating this principle, we require the following definition:

Definition 3.5. A map $\mathbb{1} \rightarrow A$ of bounded distributive lattices is a *finitely presented (fp) $\mathbb{1}$ -algebra* if it is merely equivalent to the canonical map $\mathbb{1} \rightarrow \mathbb{1}[x_1, \dots, x_n] / \langle t_1 = s_1, \dots, t_m = s_m \rangle$ for some n, m .

The definition of a homomorphism of bounded distributive lattices (a map which commutes with $0, 1, \wedge, \vee$) extends to a notion of homomorphism between fp $\mathbb{1}$ -algebras $\text{hom}_{\mathbb{1}}(A, B)$ by further requiring the underlying map to commute with the maps $\mathbb{1} \rightarrow A$ and $\mathbb{1} \rightarrow B$.

Axiom 10 (Duality). *Given an fp $\mathbb{1}$ -algebra $f : \mathbb{1} \rightarrow A$ the following map is an equivalence:*

$$\lambda a g. g(a) : A \rightarrow (\text{hom}_{\mathbb{1}}(A, \mathbb{1}) \rightarrow \mathbb{1})$$

Definition 3.6. Triangulated type theory TT_{\square} is MTT with mode theory \mathcal{M}_{\square} extended by [Axioms 1 to 10](#).

3.2 Duality and Δ^n

[Axiom 10](#) has a number of remarkable consequences for $\mathbb{1}$. While these are not specific to directed univalent universes, they allow us to construct the first non-trivial categories inside TT_{\square} . We begin with the following result—independently proven by Pugh and Sterling [[PS25](#)]*—reminiscent of various principles from synthetic domain theory.*

Lemma 3.7 (Phoa’s principle). *Evaluation at 0, 1 is an embedding $(\mathbb{1} \rightarrow \mathbb{1}) \rightarrow \mathbb{1} \times \mathbb{1}$ with image Δ^2 .*

PROOF. We first will argue via [Axiom 10](#) that $\mathbb{1}[x]$ is equivalent to $\mathbb{1} \rightarrow \mathbb{1}$ via the evaluation map. To see this, let us note that $\mathbb{1} \rightarrow \mathbb{1}[x]$ is an $\mathbb{1}$ -algebra by definition, and $\text{hom}_{\mathbb{1}}(\mathbb{1}[x], \mathbb{1}) \simeq \mathbb{1}$. Accordingly, by [Axiom 10](#), the map $\text{eval} : \mathbb{1}[x] \rightarrow (\mathbb{1} \rightarrow \mathbb{1})$ is an equivalence.

By the 2-for-3 principle of equivalences, it then suffices to show that evaluating a polynomial at 0 and 1 induces an embedding $\mathbb{1}[x] \rightarrow \mathbb{1} \times \mathbb{1}$ whose image is Δ^2 . An inductive argument allows us to conclude that $\text{eval}(p, -)$ is a monotone map from $\mathbb{1} \rightarrow \mathbb{1}$ and so evaluation of polynomials at endpoints factors through Δ^2 . We therefore are reduced to showing that this map is an equivalence. To see this, we observe that any polynomial in one variable can be placed in the following normal form: $p = \text{eval}(p, 0) \vee x \wedge \text{eval}(p, 1)$ whereby the conclusion is immediate. \square

Notation 3.8. In light of the equivalence used in the proof of Phoa’s principle, we will no longer distinguish between polynomials in one variable $\mathbb{1}[x]$ and functions $\mathbb{1} \rightarrow \mathbb{1}$.

Lemma 3.9 (Generalized Phoa’s principle).

- *The evaluation map from $\mathbb{1}^n \rightarrow \mathbb{1}$ to monotone maps $\text{Bool}^n \rightarrow \mathbb{1}$ is an equivalence.*
- *The evaluation map from $\Delta^n \rightarrow \mathbb{1}$ to monotone maps $[0 \leq \dots \leq n] \rightarrow \mathbb{1}$ is an equivalence.*

In the above, we have regarded Bool as a 2-element partial order $\text{ff} \leq \text{tt}$.

Both claims follow from induction on n and repeated application of Phoa’s principle.

Remark 3.10. The particular cube category used in our intended model of TT_{\square} is equivalent by Birkhoff duality to the category of flat finite bounded distributive lattices [[Spi16](#)]. [Lemma 3.9](#) is a manifestation of this fact.

Lemma 3.11. *$\mathbb{1}$ is simplicial.*

PROOF. To show that $\mathbb{1} \rightarrow ((i \leq j \vee j \leq i) \rightarrow \mathbb{1})$ is an equivalence, it suffices, by [Axiom 8](#), to consider $f, g : \mathbb{1}^n \rightarrow \mathbb{1}$ and show that the following is an equivalence:

$$\langle b \mid \mathbb{1}^n \rightarrow \mathbb{1} \rangle \rightarrow \langle b \mid \{\vec{x} : \mathbb{1}^n \mid f(\vec{x}) \leq g(\vec{x}) \vee g(\vec{x}) \leq f(\vec{x})\} \rightarrow \mathbb{1} \rangle$$

Using [Lemma 3.9](#), we can extend an element of the codomain to a total function $\mathbb{1}^n \rightarrow \mathbb{1}$ provided we can specify its behavior on $\vec{x} : \text{Bool}^n$. The proposition $f(\vec{x}) \leq g(\vec{x}) \vee g(\vec{x}) \leq f(\vec{x})$ holds for all $\vec{x} : \text{Bool}^n$, and so such an extension always exists and is necessarily unique. \square

Remark 3.12. A more elegant proof of the above was recently provided by Williams [[Wil25](#)].

Corollary 3.13. Δ^n is a category.

PROOF. Since there are no nontrivial invertible morphisms in Δ^n , it is trivially Rezk-complete and, as a retract of \mathbb{N}^n , it is simplicial. Therefore, it suffices to show that Δ^n is Segal.

To this end, let us consider $\Delta_1^2 \rightarrow \Delta^n$. This is equivalent to a pair of maps $f, g : \mathbb{1} \rightarrow \Delta^n$ such that $f(1) = g(0)$. Next, by the Phoa principle $f, g : \mathbb{1} \rightarrow \Delta^n$ are fully determined by n -tuples of pairs e.g., $(\pi_k(f(0)) \leq \pi_k(f(1)))_{k \leq n}$. In total then, we are given n -many 3-tuples:

$$(\pi_k(f(0)) \leq \pi_k(f(1)) = \pi_k(g(0)) \leq \pi_k(g(1)))_{k \leq n}$$

By Lemma 3.9, these are 2-simplices in Δ^n and so every horn has a unique extension as required. \square

We note that Corollary 3.13 is already a significant step forward for STT: it is the first result constructing an explicit example of a non-discrete category within the system.

3.3 Reasoning with modalities in $\mathbb{T}\mathbb{T}_{\square}$

A number of useful results in $\mathbb{T}\mathbb{T}_{\square}$ are immediate corollaries of standard results from MTT combined with one of the axioms. We record some of the most important results in this section for future use and to give a flavor for how modalities can be used to enhance simplicial reasoning.

By general results about adjoint modalities from MTT [Gra+21], we obtain the following:

Lemma 3.14.

- If $A :_{\text{id}} \mathcal{U}, B :_{\#} \mathcal{U}$ there is an equivalence $\langle \# \mid \langle b \mid A \rangle \rightarrow B \rangle \simeq (A \rightarrow \langle \# \mid B \rangle)$.
- If $A :_{\text{id}} \mathcal{U}, B :_{\text{op}} \mathcal{U}$ there is an equivalence $\langle \text{op} \mid \langle \text{op} \mid A \rangle \rightarrow B \rangle \simeq (A \rightarrow \langle \text{op} \mid B \rangle)$.

There are also dependent versions where e.g., $B :_{\#} (b \mid A) \rightarrow \mathcal{U}$.

We record two useful consequences of the transposition principle for $\text{op} \dashv \text{op}$ and $b \dashv \#$:

Lemma 3.15 (Gratzer [Gra23]). $\langle \text{op} \mid - \rangle$ commutes with colimits.

Lemma 3.16. Evaluation at endpoints $(\mathbb{1} \rightarrow \langle \# \mid A \rangle) \rightarrow (\text{Bool} \rightarrow \langle \# \mid A \rangle)$ is an equivalence.

A similar result is available for $\mathbb{1} \rightarrow -$ in light of Axiom 3:

Lemma 3.17. There is a unique map $-_{\mathbb{1}} :_{\flat} (b \mid \mathcal{U}) \rightarrow \mathcal{U}$ such that the following bijection holds:

$$\prod_{A, B :_{\flat} \mathcal{U}} \langle b \mid A^{\mathbb{1}} \rightarrow B \rangle \simeq \langle b \mid A \rightarrow B_{\mathbb{1}} \rangle$$

Consequently, $-_{\mathbb{1}}$ preserves limits. Notably, $\mathbf{1}_{\mathbb{1}} = \mathbf{1}$ and $-_{\mathbb{1}}$ commutes with taking fibers. These two facts imply that $(\sum_{A :_{\mathbb{1}} \mathcal{U}} A)_{\mathbb{1}} \rightarrow \mathcal{U}_{\mathbb{1}}$ has small fibers and induces a dependent version of this operation $-_{\mathbb{1}} : \mathcal{U}_{\mathbb{1}} \rightarrow \mathcal{U}$.

In more detail, there is a canonical functorial action of $(-)^{\mathbb{1}}$ which induces a map $(\sum_{A :_{\mathbb{1}} \mathcal{U}_i} A)_{\mathbb{1}} \rightarrow (\mathcal{U}_i)_{\mathbb{1}}$. Viewed as a family of types, we therefore obtain a map $(\mathcal{U}_i)_{\mathbb{1}} \rightarrow \mathcal{U}_{i+1}$. We can argue that this map actually factors through \mathcal{U}_i rather than just \mathcal{U}_{i+1} as follows. First, by univalence the map $\mathcal{U}_i \rightarrow \mathcal{U}_{i+1}$ is an embedding, so a factorization of $(\mathcal{U}_i)_{\mathbb{1}} \rightarrow \mathcal{U}_{i+1}$ is unique when it exists. Using Axiom 8, it suffices to check that such a factorization exists after restricting $(\mathcal{U}_i)_{\mathbb{1}} \rightarrow \mathcal{U}_{i+1}$ along some map $A :_{\flat} \mathbb{1}^n \rightarrow (\mathcal{U}_i)_{\mathbb{1}}$. Using the universal property of $(-)^{\mathbb{1}}$, we may factor A through a map $\tilde{A}_{\mathbb{1}}$ where $\tilde{A} :_{\flat} (\mathbb{1}^n)_{\mathbb{1}} \rightarrow (\mathcal{U}_i)_{\mathbb{1}}$. Since $(-)^{\mathbb{1}}$ preserves fibers, it suffices to show that the pullback P in the following is \mathcal{U}_i -small:

$$\begin{array}{ccccc} P & \longrightarrow & (\sum_{f : (\mathbb{1}^n)_{\mathbb{1}}} \tilde{A}(f))_{\mathbb{1}} & \longrightarrow & (\sum_{A :_{\mathbb{1}} \mathcal{U}_i} A)_{\mathbb{1}} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \mathbb{1}^n & \longrightarrow & ((\mathbb{1}^n)_{\mathbb{1}})_{\mathbb{1}} & \longrightarrow & (\mathcal{U}_i)_{\mathbb{1}} \end{array}$$

Finally, we note that since the ordinary version of $(-)_\mathbb{1}$ preserves \mathcal{U}_i small types, P is the pullback of \mathcal{U}_i -small types and is therefore itself \mathcal{U}_i -small.

To give an example of how these reasoning principles can be used, we show how they can be used to enhance our stock of simplicial types.

Lemma 3.18. *Given $A :_{\text{op}} \mathcal{U}$, if $\langle \text{op} \mid \text{isSimp}(A) \rangle$ then $\text{isSimp}(\langle \text{op} \mid A \rangle)$.*

PROOF. Fix $i, j : \mathbb{1}$ such that we must show $\langle \text{op} \mid A \rangle \rightarrow (i \leq j \vee j \leq i \rightarrow \langle \text{op} \mid A \rangle)$ is an equivalence. Using [Lemma 3.14](#), the codomain is equivalent to $\langle \text{op} \mid \langle \text{op} \mid i \leq j \vee j \leq i \rangle \rightarrow A \rangle$. By [Axiom 2](#) and [Lemma 3.15](#), $\langle \text{op} \mid i \leq j \vee j \leq i \rangle$ is $\neg \text{mod}_{\text{op}}(i) \geq \neg \text{mod}_{\text{op}}(j) \vee \neg \text{mod}_{\text{op}}(j) \geq \neg \text{mod}_{\text{op}}(i)$ and the conclusion follows immediately from our assumption $\langle \text{op} \mid \text{isSimp}(A) \rangle$. \square

Lemma 3.19. *If $A :_{\text{b}} \mathcal{U}$ is discrete then A is simplicial.*

PROOF. Assume A is discrete, i.e., $A \rightarrow A^\mathbb{1}$ is an equivalence. Since cubes separate by [Axiom 8](#), it suffices to show for all polynomials $p, q : \mathbb{1}[\vec{x}]$ in n variables \vec{x} that the map $A \rightarrow (\varphi(\vec{x}) \rightarrow A)$ is an equivalence, where $\varphi(\vec{x}) := p(\vec{x}) \leq q(\vec{x}) \vee q(\vec{x}) \leq p(\vec{x})$.

In turn, it suffices to give an $\mathbb{1}$ -homotopy h connecting the constant map at 0 to the identity on $\varphi(\vec{x})$, for each $\vec{x} : \mathbb{1}^n$. We notice that the straight-line homotopy $h(\vec{x}, t) = \vec{x} \wedge t$ from 0 to \vec{x} works: We have to show for each \vec{x} with $\varphi(\vec{x})$ that $\varphi(\vec{x} \wedge t)$ holds, for each t . But notice that $\varphi(0)$ is true, as any pair of constants among 0, 1 are comparable. By [Lemma 3.7](#), $\varphi(\vec{x} \wedge t)$ then holds for all t . \square

Using the adjunction $\text{b} \dashv \sharp$, we can prove that e.g., $\text{Nat} \simeq \langle \text{b} \mid \text{Nat} \rangle$ [[Gra23](#)]. Accordingly, by [Axiom 6](#):

Corollary 3.20. *Nat and Bool are both simplicial and $\mathbb{1}$ -null i.e. groupoids.*

Remark 3.21. The result analogous to [Lemma 3.19](#) for Rezk-complete Segal types does not hold, falsifying a conjecture of Weaver and Licata [[WL20](#)]. In particular, $\Delta^2 \sqcup_{\mathbb{1}} \Delta^2$ can be shown to be Rezk-complete and Segal, but is not simplicial. This same example shows that the requirement that A be annotated with b is necessary: as a family over $\mathbb{1} \times \mathbb{1}$ the type $\Delta^2 \sqcup_{\mathbb{1}} \Delta^2$ is fiberwise a proposition—explicitly, it is $\lambda i j. i \leq j \vee j \leq i$ —and therefore it is fiberwise $\mathbb{1}$ -null. If we could apply [Lemma 3.19](#) without the b -annotation we could conclude that each fiber $i \leq j \vee j \leq i$ was simplicial. Combined with the fact that $\mathbb{1} \times \mathbb{1}$ is simplicial, this leads again to the false conclusion that $\Delta^2 \sqcup_{\mathbb{1}} \Delta^2$ is simplicial.

4 The cubical spaces model

TT_{\square} is intended to be an internal language for cubical sets i.e. $\text{PSh}(\square)$ (or rather its ∞ -categorical enhancement). In order to make this precise, we construct a model of TT_{\square} in which types are realized as (families of) ∞ -presheaves over the Dedekind cube category. Immediately, we must contend with the fact that syntax is 1-categorical and models of syntax [[Gra+21](#)] are also inherently 1-categorical. To overcome this mismatch, we interpret TT_{\square} into a model category which presents the appropriate presheaf ∞ -category [[Shu15](#); [KL21](#); [Shu19](#)].

For TT_{\square} , this model category will be the *injective model structure on simplicial presheaves on Dedekind cubes* $\text{PSh}_{\text{sSet}}(\square)$. That is, types in TT_{\square} are interpreted as certain families of presheaves over \square valued in $\text{sSet} = \text{PSh}(\Delta)$. It is helpful to view the simplicial sets layer as “mixing in” homotopy theory with ordinary presheaves over \square .

The construction of the intended model of TT_{\square} is largely an exercise in combining off-the-shelf results about models of HoTT and models of MTT. In particular, Shulman [[Shu19](#)] shows that HoTT admits a model in $\text{PSh}_{\text{sSet}}(\square)$ and results of Shulman [[Shu23](#)] and Gratzer [[Gra23](#)] show that this model refines to a model of MTT. One then directly verifies that this model validates the additional

axioms required by $\mathbb{T}\mathbb{T}_{\square}$. However, for the sake of completeness we give some of the details of this process. In particular, in [Section 4.1](#) we spell out the definition of a model of MTT. In [Section 4.2](#) we show how to apply a theorem of Shulman [[Shu23](#)] to construct a model of MTT and verify that it extends appropriately to the desired model of $\mathbb{T}\mathbb{T}_{\square}$.

Remark 4.1. Models of $\mathbb{T}\mathbb{T}_{\square}$ specifically and type theory generally contain quite a lot of data to account for each connective. We will focus primarily on the specific modal connectives in $\mathbb{T}\mathbb{T}_{\square}$ in this section, as—just as in the syntax of $\mathbb{T}\mathbb{T}_{\square}$ itself—the other connectives are treated in a totally standard manner.

4.1 Models of MTT and $\mathbb{T}\mathbb{T}_{\square}$

In this subsection, we briefly recall the main aspects of the model theory of MTT. We shall not directly use these definitions in our construction of the cubical spaces model of $\mathbb{T}\mathbb{T}_{\square}$ and instead will rely on a general *coherence* result of Shulman. Accordingly, these definitions are recalled only to make the following discussion more concrete.

When originally introduced by Gratzer et al. [[Gra+21](#)], MTT was presented as a certain *generalized algebraic theory* (GAT). Consequently, the general theory of GATs ensures that there is a category of models of MTT (parameterized by the chosen mode theory) which refines the corresponding category of models of ordinary Martin-Löf type theory [[Dyb96](#)]. Let us therefore begin by recalling the definition of a model of dependent type theory:

Definition 4.2 (Awodey [[Awo18](#)]). A category with families (CwF) (\mathcal{C}, τ) consists of a category \mathcal{C} along with a morphism $\tau : U^{\bullet} \rightarrow U$ in $\text{PSh}(\mathcal{C})$ is equipped with the following chosen data:⁹

- \mathcal{C} has a chosen terminal object,
- τ is locally representable i.e., the fiber $\mathbf{y}(c) \times_U U^{\bullet}$ has a chosen representation $\mathbf{y}(c')$ for each morphism $\mathbf{y}(c) \rightarrow U$.

This structure is a model of dependent type theory with no connectives, only basic operations like context extension, variables, and substitutions. We refer the reader to Awodey [[Awo18](#)] for a careful exposition of how connectives may be integrated into this definition. Since this aspect of CwFs is carried through unchanged through the remainder of our discussion, we ignore it here.

A model of MTT elaborates on this structure by linking together CwFs via functors (which intuitively model Γ/μ). We begin with the structure necessary to model MTT without any connectives.

Definition 4.3. A model of MTT without any connectives consists of a strict 2-functor $F : \mathcal{M}^{\text{coop}} \rightarrow \text{Cat}$ along with a choice of morphism $\tau_m : U_m^{\bullet} \rightarrow U_m$ in $\text{PSh}(F(m))$ for each $m : \mathcal{M}$. We require the following additional data:

- A chosen terminal object for each $F(m)$,
- A choice of local representability structure on $F(\mu)^*(\tau_m)$ for each $\mu : m \rightarrow n$.

Note that $F(\text{id})^*(\tau_m) = \tau_m$ so each $(F(m), \tau_m)$ is a CwF. The additional requirement that each $F(\mu)^*(\tau_m)$ be locally representable is used to model *annotated* variables in a context i.e. $\Gamma, x :_{\mu} A$. Other connectives are integrated into this definition without change. For instance, the inclusion of Σ -types is accounted for by requiring that each τ_m is closed under Σ -types. This process is entirely mechanical and their interpretation in the intended model of $\mathbb{T}\mathbb{T}_{\square}$ is unsurprising, accordingly we refer the reader to Gratzer et al. [[Gra+21](#)] for further information.

⁹The fact that this is chosen data rather than a mere existence property is a quirk of generalized algebraic theories. More refined recent approaches such as those given by Uemura [[Uem21](#)] do not have this deficiency.

The central novel connective of MTT—the modal types—is slightly more complex, and so we briefly touch on their definition. We refer the reader to Gratzer et al. [Gra+21] or Gratzer [Gra23] for a careful discussion of how the following definition relates to the syntax introduced in Section 3.

Definition 4.4. A model of MTT without any connectives $(F : \mathcal{M}^{\text{coop}} \rightarrow \text{Cat}, (\tau_m)_{m:\mathcal{M}})$ supports modal types when equipped with the following data for each $\mu : n \rightarrow m$:

- A commuting square $\alpha : F(\mu)^*(\tau_n) \rightarrow \tau_m$ in $\text{PSh}(F(m))$.
- Writing $m : F(\mu)^*(U_n^\bullet) \rightarrow F(\mu)^*(U_n^\bullet) \times_{U_m} U_m^\bullet$ for the gap-map over $F(\mu)^*U_n$ induced by α , we require a *stable weak orthogonality structure* [Awo18] $s : m \pitchfork (F(\mu)^*U_n)^*(\tau_m)$ in $\text{PSh}(F(m))/F(\mu)^*U_n$.

Roughly, the first point encodes the introduction and formation rules of the μ modal type and the second encodes the elimination rule and its attendant equality.

We will not spend much time with this definition because, by a result of Shulman [Shu23], one can construct a model of MTT with all connectives from far more recognizable data. In particular, op. cit. adapts the local universes coherence construction of Lumsdaine and Warren [LW15] which promotes type-theoretic fibration categories to models.

Definition 4.5. A type-theoretic fibration category consists of a category \mathcal{C} with all finite limits with a chosen class of morphisms \mathcal{F} referred to as fibrations which satisfy the following closure conditions:

- \mathcal{F} is closed under identity and composition.
- \mathcal{F} is closed under pullback along arbitrary maps.
- \mathcal{F} is closed under pushforwards; if $f \in \mathcal{F}$ then the right adjoint to pullback along f sends fibrations to fibrations.
- Every morphism f admits a factorization $f' \circ i$ such that f' is a fibration and i is weakly left orthogonal to fibrations and this factorization is stable under pullbacks.

We refer to maps weakly left orthogonal to fibrations as *anodyne*.

Definition 4.6. A type-theoretic fibration category has a universe if there is a map $\tau : U^\bullet \rightarrow U$ such that the following conditions hold:

- Both τ and $U \rightarrow \mathbf{1}$ are fibrations.
- Fibrations arising from pulling back τ (U -small fibrations) satisfy all but the last closure condition for fibrations.
- If $f : X \rightarrow Y$ is a U -small fibration, there is a stable factorization of $\Delta_f : X \rightarrow X \times_Y X$ as $\Delta'_f \circ i$ where i is weakly orthogonal to fibrations and Δ'_f is a U -small fibration.

A hierarchy of universes is given by a collection of universes $\tau_i : U_i^\bullet \rightarrow U_i$ such that each τ_i and $U_i \rightarrow \mathbf{1}$ are U_{i+1} -fibrations.

Proposition 4.7 (Shulman [Shu23], Gratzer [Gra23]). *A model of MTT with mode theory \mathcal{M} can be constructed from the following pieces of data:*

- A pseudofunctor $F : \mathcal{M} \rightarrow \text{Cat}$ such that each $F(\mu)$ has a right adjoint G_μ ,
- For each $m : \mathcal{M}$, a choice of arrows making $F(m)$ into a type-theoretic fibration category with a hierarchy of universes.
- For each $\mu : n \rightarrow m$ and fibration $f : X \rightarrow Y$ in $F(n)$, there is a chosen stable factorization $G_\mu(f) = m[\mu]_f \circ i[\mu]_f$ such that $m[\mu]_f : X' \rightarrow G_\mu(Y)$ is a fibration and $i[\mu]_f : G_\mu(X) \rightarrow X'$ is anodyne. Additionally, we require that if $v : m \rightarrow o$ and $g : Z \rightarrow G_v(G_\mu(Y))$, the map $g^*(G_v(i[\mu]_f)) : Z \times_{G_v(G_\mu(Y))} G_v(G_\mu(X)) \rightarrow Z \times_{G_v(G_\mu(Y))} G_v(X')$ is anodyne.

In this model, contexts and types at mode m are modeled by objects and fibrations of $F(m)$. Consequently, our goal is to apply [Proposition 4.7](#) to the category of cubical spaces and to show that the resulting model of MTT satisfies the axioms required of $\mathbb{T}\mathbb{T}_{\square}$.

4.2 Constructing the cubical spaces model

First, we require the following result fundamental result from Shulman [[Shu19](#)]:

Definition 4.8. Let \square be the category of Dedekind cubes i.e., the full subcategory of Cat generated by finite products of the category $[1] = \{0 \rightarrow 1\}$.

Proposition 4.9 (Shulman [[Shu19](#)]). *$\text{PSh}_{\text{sSet}}(\square)$ and the collection of injective fibrations forms a type-theoretic fibration category with a hierarchy of univalent universes.*

We will now use this type-theoretic fibration category as the basic input for [Proposition 4.7](#). In particular, we consider the functor $F : \mathcal{M}_{\square} \rightarrow \text{Cat}$ which sends the unique object m to $\text{PSh}_{\text{sSet}}(\square)$ and interprets the 1-cells as follows:

$$F(b)(X) = [n] \mapsto X([0]) \quad F(\text{op})(X) = [n] \mapsto X([n]^{\text{op}}) \quad F(\sharp)(X) = [n] \mapsto X([0])^n$$

Here we have written $-^{\text{op}}$ for the unique functor $\square \rightarrow \square$ which sends $[n]$ to $[n]$ but exchanges 0 for 1. We may directly check that all of the required equalities are satisfied and it is also clear that $F(b) \dashv F(\sharp)$. It remains to define F on the generating inequalities $b \leq \text{id}$ and $\text{id} \leq \sharp$. We realize these by the unique morphism in $\text{hom}_{\square}([n], [0])$ (the counit of the aforementioned adjunction) and its transpose (the unit).

In order to show that this functor is well-defined, we must show that it sends all 2-cells $\alpha : \mu \rightarrow \nu$ to the same natural transformations $F(\alpha) : F(\mu) \rightarrow F(\nu)$. For instance, we must argue that the two 2-cells $b \circ b \rightarrow b$ induced by $b \leq \text{id}$ and whiskering on either side are sent to the same natural transformation. A priori, this is far from obvious: the presence of various equalities between modalities in \mathcal{M}_{\square} allows for non-obvious 2-cells. In this particular model, however, our task is far easier. The fact that \square , sSet , and Set all have trivial centers ensures that $\text{PSh}_{\text{sSet}}(\square)$ has a trivial center and we prove the following:

Lemma 4.10. *For each $\mu, \nu : m \rightarrow m$ in \mathcal{M}_{\square} , there is at most one natural transformation $F(\mu) \rightarrow F(\nu)$.*

PROOF. First, we note that the equations for \mathcal{M}_{\square} ensure that we need only consider $\mu, \nu \in \{\text{op}, \sharp, b, \text{id}\}$. Further, by adjointness, we may ignore the cases where $\nu \in \{\sharp, \text{op}\}$. We check the remaining cases directly.

For instance, if $\mu = \nu = \text{id}$, then we note that a natural transformation $F(\mu) \rightarrow F(\nu)$ is determined by its behavior at $\mathbf{y}([1])$; both sides preserve products, tensoring by simplicial sets, and colimits and all cubical spaces are generated from $\mathbf{y}([1])$ under these operations. However, there is only one morphism $[1] \rightarrow [1]$ which preserves endpoints: the identity map. The same reasoning rules out the possibility of any natural transformations with $\mu = \text{id}, \nu = \text{op}$ or $\mu = \text{id}, \nu = b$. It also guarantees that there is exactly one natural transformation $F(\text{id}) \rightarrow F(\sharp)$.

The case where $\mu = b$ is similar, so we focus on the remaining case where $\mu = \sharp$. This case is slightly more complex, since $F(\sharp)$ does not preserve colimits and so we can immediately reduce to checking behavior at $\mathbf{y}([1])$. However, we can immediately rule out natural transformations $F(\sharp) \rightarrow F(\text{id})$, $F(\sharp) \rightarrow F(\text{op})$, or $F(\sharp) \rightarrow F(b)$ by considering the behavior of such morphisms at $\mathbf{y}([1])$. To show that there is exactly one morphism $F(\sharp) \rightarrow F(\sharp)$, we note that it suffices to show that there is exactly one natural transformation $\alpha : [0]^* \rightarrow [0]^*$ as functors from $\text{PSh}_{\text{sSet}}(\square) \rightarrow \text{sSet}$. Now these functors again preserve limits, colimits, and tensoring by simplicial sets and so we reduce to considering the behavior of α on $[0]$ and $[1]$, where we see it must once more be the identity. \square

With this well-defined functor $F : \mathcal{M}_{\boxtimes} \rightarrow \text{Cat}$, we immediately check that each 1-cell in \mathcal{M}_{\boxtimes} is sent to a right adjoint and so there is a conjugate functor $\bar{F} : \mathcal{M}_{\boxtimes}^{\text{coop}} \rightarrow \text{Cat}$ and it is to this we apply [Proposition 4.7](#). The first requirement of this theorem (that each 1-cell is sent by \bar{F} to a right adjoint) is automatic in this case. The previously cited result of Shulman already shows that $F(m)$ has an appropriate choice of type-theoretic model structure. It remains only to handle the third point, which governs the interpretation of modalities into this putative model. For three of the four modalities we must interpret, this is trivial:

Lemma 4.11. *For $\mu \in \{\sharp, \text{op}, \text{id}\}$, $F(\mu)$ is a right Quillen functor for both the injective and projective model structures.*

In particular, given a fibration $f : X \rightarrow Y$ in $\bar{F}(m)$, we may trivially factorize e.g., $F(\sharp)(X) \rightarrow F(\sharp)(Y)$ into an anodyne map (trivial injective cofibration) followed by an (injective) fibration, simply by taking the identity map followed by $F(\sharp)(f)$. This factorization satisfies all the desiderata of the third point of [Proposition 4.7](#) trivially. Unfortunately, $F(b)$ is not a right Quillen functor for the injective model structure and a more elaborate approach required in this case.

Lemma 4.12. *Given an injective fibration $p : X \rightarrow Y$, there is a factorization of $F(b)(f)$ into a trivial cofibration i followed by an injective fibration f such that (1) this factorization is stable under pullback and (2) if $v : m \rightarrow m$ in \mathcal{M}_{\boxtimes} and $z : Z \rightarrow F(v)(F(\mu)(Y))$ then $z^*(F(v)(i))$ is a trivial cofibration.*

PROOF. We begin by noting that while $p' = F(b)(p)$ is not necessarily an injective fibration, it is a projective fibration. Indeed, p is a projective fibration so that, by definition, $p([0])$ is a fibration and so p' is a levelwise (i.e., projective) fibration. We may now apply the cobar construction detailed by Shulman [[Shu19](#), Definition 8.17] to obtain a stable factorization of p' into a trivial cofibration i followed by an injective fibration f , as required.

For the second condition, we note that p' and f are both projective fibrations and, consequently, so too are $F(v)(p')$ and $F(v)(f)$. Moreover, since injective trivial cofibrations are precisely a levelwise trivial cofibration, we may also conclude that $F(v)(i)$ is an injective trivial cofibration—note that trivial cofibrations are closed under cartesian products in sSet for the case where $v = \sharp$. Finally, since pullbacks are computed levelwise and the model structure on sSet is right proper, we see that $z^*(F(v)(i))$ remains a trivial cofibration for all $z : Z \rightarrow F(v)(F(\mu)(Y))$. \square

Lemma 4.13. *There is a model of MTT with mode theory \mathcal{M}_{\boxtimes} in $\text{PSh}_{\text{sSet}}(\square)$ which interprets modalities using the functors described by F*

Finally, we must show that this model of MTT validates the axioms necessary for TT_{\boxtimes} . [Axiom 4](#) is an immediate consequence of Shulman [[Shu19](#)]. Notably, Gratzer [[Gra23](#)] shows that [Axiom 5](#) holds for \sharp , op , and id —since they are *dependent right adjoints*—and that it holds for b since it is an internal left adjoint.

[Axioms 1, 2, 7](#) and [10](#) are all statements about sets—0-truncated types—and so hold in the above model if and only if they hold in the ordinary interpretation of type theory in $\text{PSh}(\square)$. That is, for these axioms we may ignore the simplicial dimension of $\text{PSh}_{\text{sSet}}(\square)$. In this case, each of these except [Axiom 10](#) is a routine verification.¹⁰ The duality axiom, finally, follows from a result of Blechschmidt [[Ble23](#), Theorem 4.11] combined with the following lemma:

Lemma 4.14. *There is a canonical geometric embedding $\text{PSh}(\square) \rightarrow \text{Set}[\text{DLat}]$ of cubical sets into the classifying topos of distributive lattices. Under this embedding, $\mathbf{y}([1])$ in $\text{PSh}(\square)$ is sent to the generic bounded distributive lattice.*

¹⁰See also [[Mye25](#)] for a general topos-theoretic account.

The remaining axioms (Axioms 3, 6, 8 and 9) are verified by straightforward (if tedious) categorical arguments. In each case, we must show that a certain function in type theory is an equivalence. After unfolding to the model, this amounts to showing that a certain morphism is a weak equivalence. In each case, these statements admit natural ∞ -categorical interpretations (e.g., Axiom 3 postulates the existence of an amazing right adjoint to $(-)^{\flat}$), but a model-categorical argument is necessary to connect this fact to the interpretation of $\mathbb{T}\mathbb{T}_{\square}$ in $\text{PSh}_{\text{sSet}}(\square)$. Since these arguments are nearly all the same, we choose to focus on the most complex: Axiom 3.

Lemma 4.15. *The following type is inhabited in the model of MTT in $\text{PSh}_{\text{sSet}}(\square)$:*

$$(A \cdot_{\flat} \mathcal{U}) \rightarrow \sum_{A_{\flat} \cdot_{\flat} \mathcal{U}} \sum_{e_{\flat} : (A_{\flat})^{\flat} \rightarrow A} \prod_{B \cdot_{\flat} \mathcal{U}} \text{isEquiv}(\langle b \mid B \rightarrow A_{\flat} \rangle \rightarrow \langle b \mid B^{\flat} \rightarrow A \rangle)$$

PROOF. For concision, let us denote the codomain of the above function by $\text{isRepr}(A)$. First, we note that we may prove directly in MTT that $\text{isRepr}(A)$ is a proposition, so this type is inhabited if and only if the following fibration is a trivial fibration in our model:

$$\llbracket \sum_{\text{mod}_{\flat}(A) : \langle b \mid \mathcal{U} \rangle} \text{isRepr}(A) \rrbracket \rightarrow \llbracket \langle b \mid \mathcal{U} \rangle \rrbracket$$

To prove this, we may restrict our attention to showing that the restriction of this fibration along the trivial cofibration $\text{Const}(\llbracket \mathcal{U} \rrbracket([0])) \rightarrow \llbracket \langle b \mid \mathcal{U} \rangle \rrbracket$ is trivial. Here $\text{Const}(X)$ denotes the constant cubical space with value X .

It suffices to check that this morphism is a trivial fibration fiberwise. We therefore need to show that the restriction of this map along $A : \mathbf{y}([n]) \rightarrow \text{Const}(\llbracket \mathcal{U} \rrbracket([0]))$ is a trivial fibration. However, A factors through $\mathbf{y}([0])$. Consequently, we may restrict our attention to the case where $n = 0$.

Accordingly, we may fix $X \in \llbracket \mathcal{U} \rrbracket([0])$ i.e. an injectively fibrant presheaf X and we must show that $\llbracket \text{isRepr}(A) \rrbracket_{A \rightarrow X}$ is inhabited. We now unfold isRepr to see that it suffices to construct another injectively fibrant presheaf Y such that for all fibrant Z , there is a natural weak equivalence $\text{hom}(Z^{\mathbf{y}([1])}, X) \rightarrow \text{hom}(Z, X')$.

Let us now note that $(-)^{\mathbf{y}([1])} \cong (- \times [1])^*$. This is a left adjoint which preserves injective cofibrations and therefore has a right adjoint $(-)^{\mathbf{y}([1])}$ which preserves injective fibrations. We choose $X' = (X)^{\mathbf{y}([1])}$ and the rest follows. \square

Theorem 4.16. *$\mathbb{T}\mathbb{T}_{\square}$ has a model in $\text{PSh}_{\text{sSet}}(\square)$ where types are injective fibrations and modalities are interpreted as described above.*

Crucially, within this model simplicial types are precisely those belonging to the subtopos $\text{PSh}_{\text{sSet}}(\Delta)$ [SW21]. Consequently, the adequacy result from Riehl and Shulman [RS17] applies and we conclude that this model shows that any fact proven about categories and groupoids inside of $\mathbb{T}\mathbb{T}_{\square}$ is a valid proof for the standard definition of ∞ -categories.

Theorem 4.17. *Categories in $\mathbb{T}\mathbb{T}_{\square}$ adequately model ∞ -categories.*

5 Covariant and amazingly covariant families

In Section 2, we saw how groupoids were defined internally as those types satisfying $\text{isGroupoid } A = \text{isEquiv}(A \rightarrow A^{\flat})$. We might hope this induces a directed univalent universe of groupoids directly, by considering $\mathcal{U}_{\text{grp}} = \sum_{A : \mathcal{U}} \text{isGroupoid } A$. However, this is far from our desired universe. Most glaringly, while $F : A \rightarrow \mathcal{U}_{\text{grp}}$ is a family of groupoids over A , this family is not required to respect the category structure of A in any way. In fact, one may show that a map $F : \mathbb{1} \rightarrow \mathcal{U}_{\text{grp}}$ is akin to an unstructured relation between $F(0)$ and $F(1)$ and nothing like the function required for directed univalence.

Example 5.1. By assumption, $\mathbb{1}$ is a set and so $f = \lambda i. i = 0$ is a function $\mathbb{1} \rightarrow \mathbf{HProp}$. Since each proposition is a groupoid, this ensures that f factors through \mathcal{U}_{grp} despite the fact that there can be no function from $f(0) \rightarrow f(1)$.

In order to rectify this and define \mathcal{S} , we shall require a theory of families of groupoids where a morphism $f : \text{hom}(a, a')$ in A induces a functor of groupoids $F(a) \rightarrow F(a')$. Riehl and Shulman [RS17] termed these *covariant families* and they are further studied by Buchholtz and Weinberger [BW23]. As mentioned in the introduction, we shall also require a modal version of covariant families $F : A \rightarrow \mathcal{U}$ which are covariant not only in A but also in the entire context.

5.1 Covariant families and transport

We begin by recalling the definition of a covariant family from Riehl and Shulman [RS17].

Definition 5.2. A family $A : X \rightarrow \mathcal{U}$ is *covariant* if the following proposition holds:

$$\text{isCov}(A) = \prod_{x:\mathbb{1} \rightarrow X} \prod_{a_0:A(x_0)} \text{isContr}(\sum_{a_1:A(x_1)} \text{hom}_A^x(a_0, a_1))$$

Convention 5.3. While not strictly necessary, we will assume that the base of a covariant family A is a Segal type unless explicitly noted otherwise.

We recall a few facts about covariant families (also due to Riehl and Shulman [RS17]).

Lemma 5.4. *If $A : X \rightarrow \mathcal{U}$ is covariant and $f : Y \rightarrow X$ then $A \circ f$ is also covariant.*

Lemma 5.5. *Given $\phi : \text{isCov}(A : X \rightarrow \mathcal{U})$ and $f : \text{hom}(x_0, x_1)$ then there is an induced transport map $\text{coe}_{A \circ f} : A(x_0) \rightarrow A(x_1)$. Moreover, transport maps respect composition and identities.*

PROOF SKETCH. One defines the transport map $\text{coe}_{A \circ f}(a_0) = \pi_1(\phi f a_0)$. We leave it to the reader to check that this has the appropriate type and that the expected identities are satisfied. \square

Lemma 5.6. *Given $\phi : \text{isCov}(A : X \rightarrow \mathcal{U})$ and $x : X$, the fiber $A(x)$ is a groupoid.*

PROOF. Since covariant families stable under base-change, $A(x)$ is a covariant family $\mathbf{1} \rightarrow \mathcal{U}$. Unfolding definitions, we conclude that the following holds:

$$(a : A(x)) \rightarrow \text{isContr}(\sum_{f:\mathbb{1} \rightarrow A(x)} f(0) = a)$$

This is equivalent to the proposition that $\text{eval}(0) : (\mathbb{1} \rightarrow A(x)) \rightarrow A(x)$ is an equivalence. By 3-for-2, this implies that $\text{const} : A(x) \rightarrow A(x)^{\mathbb{1}}$ is an equivalence, as $\text{eval}(0) \circ \text{const} = \text{id}$. \square

It is often helpful to rephrase covariant families in terms of orthogonality conditions:

Definition 5.7. Given a type family $A : X \rightarrow \mathcal{U}$, we shall write \tilde{A} for the *total type* $\sum_{x:X} A(x)$.

Lemma 5.8 ([RS17, Theorem 8.5]). *A family $A : X \rightarrow \mathcal{U}$ is covariant if and only if the projection map $\tilde{A} \rightarrow X$ is right orthogonal to $\{0\} \hookrightarrow \mathbb{1}$ i.e., if $(\tilde{A})^{\mathbb{1}} \rightarrow (\tilde{A})^{\{0\}} \times_{X^{\{0\}}} X^{\mathbb{1}}$ is an equivalence.*

This formulation also makes plainer the fact that covariant families are the simplicial type theory analogue of left fibrations in ordinary ∞ -category theory [Joy08; Lur09].

Finally, using the characterization of covariance as an orthogonality condition, we are able to prove the following:

Lemma 5.9. *If $A : \mathbb{b} X \rightarrow \mathcal{U}$ is covariant then A is simplicial i.e., it factors through $\mathcal{U}_{\square} \rightarrow \mathcal{U}$.*

PROOF. By **Axiom 8**, it suffices to show the following:

$$\text{isEquiv}(\langle \mathbb{b} \mid \mathbb{1}^n \rightarrow \sum_{x:X} \text{isSimp}(A(x)) \rangle \rightarrow \langle \mathbb{b} \mid \mathbb{1}^n \rightarrow X \rangle)$$

In particular, we may assume that $X = \mathbb{1}^n$ by restricting A and so we hereafter also assume that X is simplicial.

With this in mind, A factors through \mathcal{U}_{\square} if and only if $\sum_{x:X} A(x)$ is simplicial. Next, we observe that $\sum_{x:X} A(x)$ is simplicial if and only if the projection map $\sum_{x:X} A(x) \rightarrow X$ is right orthogonal to $(\sum_{i,j:\mathbb{1}} i \leq j \vee j \leq i) \rightarrow \mathbb{1} \times \mathbb{1}$.

Since A and X are both b -annotated, we may use [Axiom 8](#) again to reduce to showing that $\sum_{x:X} A(x) \rightarrow X$ is b -orthogonal to $\mathbb{1}^n \times (\sum_{i,j:\mathbb{1}} i \leq j \vee j \leq i) \rightarrow \mathbb{1}^{n+2}$. We will now argue that this map is in the left class generated by the inclusion $\{0\} \rightarrow \mathbb{1}^n$; we know that $\sum_{x:X} A(x) \rightarrow X$ is right orthogonal to such maps by virtue of our assumption that A was covariant.

To this end, we consider the canonical inclusions $\{0\} \rightarrow \mathbb{1}^n \times (\sum_{i,j:\mathbb{1}} i \leq j \vee j \leq i)$ and $\{0\} \rightarrow \mathbb{1}^{n+2}$. Using the 3-for-2 property available for the left class of maps, to show that $\sum_{x:X} A(x) \rightarrow X$ is right orthogonal to $\mathbb{1}^n \times (\sum_{i,j:\mathbb{1}} i \leq j \vee j \leq i) \rightarrow \mathbb{1}^{n+2}$, it suffices to show that (1) it is orthogonal to $\{0\} \rightarrow \mathbb{1}^{n+2}$ and (2) it is orthogonal to $\{0\} \rightarrow \mathbb{1}^n \times (\sum_{i,j:\mathbb{1}} i \leq j \vee j \leq i)$. The first claim is immediate from our assumption that A is covariant.

For the second claim, we note that $\{0\} \rightarrow \mathbb{1}^n \times (\sum_{i,j:\mathbb{1}} i \leq j \vee j \leq i)$ is the pushout of the maps $\{0\} \rightarrow \mathbb{1}^n \times \mathbb{1}$ and $\{0\} \rightarrow \mathbb{1}^n \times \Delta^2$. We note that the latter is a retract of $\{0\} \rightarrow \mathbb{1}^n \times \mathbb{1}^2$ and so both of these maps are orthogonal to $\sum_{x:X} A(x) \rightarrow X$, again by our assumption that A is covariant. The conclusion follows by the closure of left classes of maps under colimits. \square

5.2 Amazing covariance

We now refine our search from a universe of groupoids to a universe of *covariant fibrations*. That is, we wish to define some universe \mathcal{S} such that a map $A \rightarrow \mathcal{S}$ corresponds (in some sense) to a covariant fibration over A . Let us leave this correspondence imprecise for now and consider the behavior of \mathcal{S} .

In light of [Lemma 5.6](#), the points of \mathcal{S} will be covariant over 1 i.e. groupoids. However, elements $f : \mathbb{1} \rightarrow \mathcal{S}$ will become richer: they are covariant fibrations $B \rightarrow \mathbb{1}$, therefore consist not only of a pair of groupoids B_0, B_1 over 0 and 1 , but also include a transport function $B_0 \rightarrow B_1$ ([Lemma 5.5](#)). Phrased differently, a homomorphism $F : \mathbb{1} \rightarrow \mathcal{S}$ contains an ordinary function $F(0) \rightarrow F(1)$.

Clearly this is a step towards directed univalence over $\sum_{A:\mathcal{U}} \text{isGroupoid } A$, but it is far from obvious how to define such a type \mathcal{S} . In particular, while we have sketched how behavior ought to differ between elements of \mathcal{S} compared with functions $\mathbb{1} \rightarrow \mathcal{S}$ and so on, we cannot really cleanly divide elements of \mathcal{S} from functions into \mathcal{S} within type theory. Within dependent type theory, every element of \mathcal{S} is formed to some context Γ and if that Γ happens contains a variable $i : \mathbb{1}$, then this term will induce a function $\mathbb{1} \rightarrow \mathcal{S}$.

There is an even more straightforward way to see why this causes a problem. Suppose we attempt to define another subtype of \mathcal{U} to isolate this universe of covariant fibrations $\sum_{A:\mathcal{U}} \text{isCov}(A)$. A cursory inspection reveals this to be nonsense: being covariant is not a property of A , it is a property of a family of types $A : X \rightarrow \mathcal{U}$. So in this ‘definition’, what exactly is A covariant over?

It is here that modalities are vital: A should be covariant with respect to the entire ambient context. This is not something that can be expressed in standard type theory, but with the amazing right adjoint to $\mathbb{1} \rightarrow -$ we are able to define such a subtype.

Types covariant over Γ . We define a predicate on types $\text{isACov} : \mathcal{U} \rightarrow \text{HProp}$ which encodes whether a type is covariant over the entire context following Riley [[Ril24](#)]. We note that this predicate is a refinement of Licata et al. [[Lic+18](#)] which capitalizes on the existence of the amazing right adjoint as a proper modality. The construction of this predicate proceeds in two steps:

- (1) We begin by observing that isCov_X has the type $(X \rightarrow \mathcal{U}) \rightarrow \text{HProp}$ for each type $X : \mathcal{U}$. In particular, $\text{isCov}_{\mathbb{1}} : (\mathbb{1} \rightarrow \mathcal{U}) \rightarrow \text{HProp}$

- (2) As isCov_{\parallel} , we may apply [Lemma 3.17](#) to obtain a function $\mathcal{U} \rightarrow \text{HProp}_{\parallel}$.
- (3) Finally, we post-compose with the dependent version of $(-)\parallel$ to construct a function $\mathcal{U} \rightarrow \text{HProp}$.

All told, we obtain a predicate $\text{isACov} : \mathcal{U} \rightarrow \text{HProp}$ which encodes whether a given type is covariant over the entire context.

Definition 5.10. A type is said to be *amazingly covariant* when it satisfies isACov .

We begin by substantiating the claim that $\text{isACov}(A)$ implies that A is truly covariant over all variables in the context.

Lemma 5.11. *Given $F : X \rightarrow \sum_{A:\mathcal{U}} \text{isACov}(A)$, the type family $F_0 = \pi_1 \circ F$ is covariant.*

Notation 5.12. We will write $\mathcal{U}_{\text{ACov}}$ for the subtype $\sum_{A:\mathcal{U}} \text{isACov}(A)$.

PROOF. We must show $\text{isCov}(F_0)$. First, we note that since $X \rightarrow \mathcal{U}$ being covariant implies that the composite $Y \rightarrow X \rightarrow \mathcal{U}$ is covariant for all $Y \rightarrow X$, we may reduce to the generic case where $X = \mathcal{U}_{\text{ACov}}$ and, in particular, where F is \flat -annotated.

Next, note that $\mathcal{U}_{\text{ACov}}$ fits into the following pullback diagram:

$$\begin{array}{ccc}
 \mathcal{U}_{\text{ACov}} & \longrightarrow & \mathbf{1}_{\parallel} \\
 \downarrow & & \downarrow \\
 \mathcal{U} & \xrightarrow{\text{isCov}^{\leftarrow}} & \text{HProp}_{\parallel}
 \end{array}$$

We therefore note that $X \rightarrow \mathcal{U}_{\text{ACov}}$ is equivalent to asking for a pair of maps $F_0 : X \rightarrow \mathcal{U}$ and $F_1 : X \rightarrow \mathbf{1}_{\parallel}$ such that the induced maps $X \rightarrow \text{HProp}_{\parallel}$ agree. Since $X \rightarrow \mathcal{U}_{\text{ACov}}$ along with the above pullback diagram consists only of \flat -annotated objects, we may therefore transpose to conclude that the induced maps $X^{\flat} \rightarrow \text{HProp}$ agree. Unfolding, these maps are given as follows:

$$\begin{aligned}
 \lambda f : X^{\flat}. \text{isCov}(F_0 \circ f) \\
 \lambda f : X^{\flat}. \mathbf{1}
 \end{aligned}$$

Consequently, that these two maps agree amounts to a proof that F_0 is covariant, as required. \square

We emphasize that in the above $\mathcal{U}_{\text{ACov}}$ does not “know about” X . In particular, this is a subtype of \mathcal{U} such that any map into this subtype induces covariant families.

Finally, the additional burden of being covariant over the context does not apply when working under $\langle b \mid - \rangle$, a reflection of the fact that $\langle b \mid A \rangle$ is “a proof of A not depending on the context.”

Lemma 5.13. *If $X :_{\flat} \mathcal{U}$ and $A :_{\flat} X \rightarrow \mathcal{U}$ then $\langle b \mid (x : X) \rightarrow \text{isACov}(A(x)) \rangle = \langle b \mid \text{isCov}(A) \rangle$.*

PROOF. Since both isACov and isCov are propositions, it suffices to construct a bi-implication. First, let us suppose that $z :_{\flat} (x : X) \rightarrow \text{isACov}(A(x))$ holds. Applying the introduction rule for $\langle b \mid - \rangle$, we wish to show that $\text{isCov}(A)$ holds. Let us note that z implies that $A : X \rightarrow \mathcal{U}$ factors through $\mathcal{U}_{\text{ACov}}$. That is, we have a diagram of the following shape (consisting of \flat -types and functions):

$$\begin{array}{ccccc}
& & \mathcal{U}_{\text{ACov}} & \longrightarrow & \mathbf{1}_{\parallel} \\
& \nearrow & \downarrow & & \downarrow \\
X & \xrightarrow{A} & \mathcal{U} & \xrightarrow{\text{isCov}^{\leftarrow}} & \text{HProp}_{\parallel}
\end{array}$$

By the naturality of transposition, we conclude that $\text{isCov} \circ A_* = \lambda_{\cdot} \cdot 1$ as functions from $X^{\parallel} \rightarrow \text{HProp}$. Consequently, A is covariant as required.

For the reverse direction, suppose that $z :_{\parallel} \text{isCov}(A)$. It suffices to construct a (necessarily unique) function $A_0 :_{\parallel} X \rightarrow \mathcal{U}_{\text{ACov}}$ with an identification $\pi \circ A_0 = A$. By the universal property of pullbacks along with the identification $\mathbf{1} = \mathbf{1}_{\parallel}$, it therefore suffices to construct an identification between $\text{isCov}_{\parallel} \circ A$ and $\lambda_{\cdot} \cdot \mathbf{1}_{\parallel}$. After transposing, we therefore must show that the map $X^{\parallel} \rightarrow \text{HProp}$ sending $x : \parallel \rightarrow X$ to $\text{isCov}(A \circ x)$ is equal to the map sending $x : \parallel \rightarrow X$ to $\mathbf{1}$. This, finally, follows immediately from our assumption that $\text{isCov}(A)$ holds. \square

5.3 Closure properties of amazing covariance

Given the strength of isACov , the reader may wonder how one ever proves that $\text{isACov}(A)$ for any element $A : \mathcal{U}$. In this section, we give a partial answer by building up a stock of amazingly covariant types. We shall see in [Section 6](#) that these results undergird the closure properties of our directed univalent universe. Our main result is the following:

Lemma 5.14. *In what follows, let us assume that $A, A_0, A_1 : \mathcal{U}$ and $B : A \rightarrow \mathcal{U}$.*

- (1) *If $X :_{\parallel} \mathcal{U}$ then $\text{isACov}(\langle b \mid X \rangle)$.*
- (2) *If $i : \parallel$ then $\text{isACov}(i = 1)$.*
- (3) *If $\text{isACov}(A)$ and $a, b : A$ then $\text{isACov}(a = b)$.*
- (4) *If $\text{isACov}(A)$ and $(a : A) \rightarrow \text{isACov}(B(a))$ then $\text{isACov}(\sum_{a:A} B(a))$.*
- (5) *If $\text{isACov}(A_0)$, $\text{isACov}(A_1)$ and $f, g : A_0 \rightarrow A_1$ then $\text{isACov}(\text{Coeq}(f, g))$.¹¹*

Moreover, isACov is closed under Π -types provided modalities are used to manage the variance swap:

- (6) *If $C :_{\text{op}} \mathcal{U}$ and $D : (\text{op} \mid A) \rightarrow \mathcal{U}$ such that $\langle \text{op} \mid \text{isACov}(C) \rangle$ and $(c :_{\text{op}} C) \rightarrow \text{isACov}(D(c))$ then $\text{isACov}((c :_{\text{op}} C) \rightarrow D(c))$.*

We record a useful special case of (6) which follows from the involutive property of $\langle \text{op} \mid - \rangle$:

Corollary 5.15. *If $X :_{\parallel} \mathcal{U}$, $B : X \rightarrow \mathcal{U}$ such that $\prod_{x:X} \text{isACov}(B(x))$ then $\text{isACov}(\prod_{x:X} B(x))$.*

We limit ourselves to proving three representative cases of the above theorem: (2), (4), (6). These are indicative of the remaining cases (and those we have particular use for in [Section 6](#)).

Lemma 5.16. *If $i : \parallel$ then $\text{isACov}(i = 1)$.*

PROOF. To prove this result, we shall switch to a more general goal, $\langle b \mid (i : \parallel) \rightarrow \text{isACov}(i = 1) \rangle$, which can then be specialized to yield the original result. Using [Lemma 3.17](#), it suffices to construct an element of $\langle b \mid (f : \parallel \rightarrow \parallel) \rightarrow \text{isCov}(\lambda j. f(j) = 1) \rangle$

Since we have no additional hypotheses in this proof, we may forget the $\langle b \mid - \rangle$ and assume $f : \parallel \rightarrow \parallel$. By [Axiom 1](#), \parallel is an h-set and so $\text{isCov}(f(j) = 1)$ is equivalent to showing that $f(0) = 1$ implies that $f(1) = 1$ i.e. that f is monotone. This is an immediate consequence of [Lemma 3.7](#). \square

Lemma 5.17. *If $\text{isACov}(A)$ and $(a : A) \rightarrow \text{isACov}(B(a))$ then $\text{isACov}(\sum_{a:A} B(a))$.*

¹¹Here $\text{Coeq}(f, g)$ denotes the coequalizer of f, g realized as a higher-inductive type [\[Uni13\]](#).

PROOF. As before, we begin by generalizing slightly and instead proving the following:

$$\langle b \mid ((A, B) : \sum_{A: \mathcal{U}_{\text{ACov}}} \mathcal{U}_{\text{ACov}}^A \rightarrow \text{isACov}(\sum_{a:A} B(a))) \rangle$$

Let us begin by applying (the dependent version of) [Lemma 3.17](#) such that it suffices to show that the following holds instead:

$$\langle b \mid ((A, B) : \sum_{A: \mathcal{U}_{\text{ACov}}^{\mathbb{1}}} (i : \mathbb{1}) \rightarrow A(i) \rightarrow \mathcal{U}_{\text{ACov}}) \rightarrow \text{isCov}(\lambda i. \sum_{a:A} B(i, a)) \rangle$$

Since we have no additional assumptions, we may drop the b and assume we are given $A : \mathcal{U}_{\text{ACov}}^{\mathbb{1}}$ and $B : (i : \mathbb{1}) \rightarrow A i \rightarrow \mathcal{U}_{\text{ACov}}$. In light of [Lemma 5.11](#), we note that A is covariant and, moreover, so too is $\lambda i. B i (a i)$ for any $a : (i : \mathbb{1}) \rightarrow A i$.

In total then, we are reduced to proving the following: if $A : \mathbb{1} \rightarrow \mathcal{U}$ and $B : (i : \mathbb{1}) \rightarrow A(i) \rightarrow \mathcal{U}$ such that $\text{isCov}(A)$ and $\text{isCov}(\lambda i. B i (a i))$ then $\text{isCov}(\lambda i. \sum_{a:A} B i a)$. This statement is proven by Buchholtz and Weinberger [[BW23](#), Proposition 6.2.1]. \square

For clarity, we show the proof of the non-dependent version of (6). This is all that is required in the next section and illustrates the core idea with less noise from handling indices.

Lemma 5.18. *If $C :_{\text{op}} \mathcal{U}_{\text{ACov}}$ and $D : \mathcal{U}_{\text{ACov}}$ then $\text{isACov}(\langle \text{op} \mid C \rangle \rightarrow D)$.*

PROOF. Following the previous two arguments, we will begin by proving this in a b -context to deal with isACov . That is, we first prove the following:

$$\langle b \mid (C :_{\text{op}} \mathcal{U}_{\text{ACov}})(D : \mathcal{U}_{\text{ACov}}) \rightarrow \text{isACov}(\langle \text{op} \mid C \rangle \rightarrow D) \rangle$$

Arguing as in [Lemma 5.17](#), we may use [Lemma 5.11](#) and [Lemma 3.17](#) to assume that we are given $C :_{\text{op}} \mathbb{1} \rightarrow \mathcal{U}$ and $D : (i : \mathbb{1}) \rightarrow \mathcal{U}$ such that (1) $\langle \text{op} \mid \text{isCov}(C) \rangle$ and (2) $\text{isCov}(D)$ and such that we must show (3) that $\text{isCov}(\lambda i. \langle \text{op} \mid C(-i) \rangle \rightarrow D(i))$.

We begin by noting that $\langle \text{op} \mid \text{isCov}(C) \rangle$ is equivalent to the following assumption:

$$(c_1 : \langle \text{op} \mid C(0) \rangle) \rightarrow \text{isContr}(\sum_{c:(i:\mathbb{1}) \rightarrow \langle \text{op} \mid C(-i) \rangle} c(1) = c_1)$$

Unfolding our obligation, we must show that given $f_0 : \langle \text{op} \mid C(1) \rangle \rightarrow D(0)$ that the following type is contractible:

$$\sum_{f:(i:\mathbb{1}) \rightarrow \langle \text{op} \mid C(-i) \rangle \rightarrow D(i)} f(0) = f_0$$

We first informally describe how one produces a center of contraction. Given $i : \mathbb{1}$ and $c : \langle \text{op} \mid C(-i) \rangle$, we use the assumption that C is op -covariant as formulated above to construct a (unique) function $\bar{c} : (j : \mathbb{1}) \rightarrow \langle \text{op} \mid C(-(i \wedge j)) \rangle$ such that $\bar{c}(1) = c$. We then observe that $\bar{c}(0) : \langle \text{op} \mid C(1) \rangle$ and so $d_0 = f_0(\bar{c}(0)) : D(0, \bar{c}(0))$. Extending this d_0 to a line along $D(i \wedge -)$ gives the line $\bar{d} : (j : \mathbb{1}) \rightarrow D(j \wedge i)$ and we choose $f(i, c) = f_0(\bar{d}(1)) : (c : \langle \text{op} \mid C(-i) \rangle) \rightarrow D(i, c)$. If $i = 0$, \bar{c} and \bar{d} are canonically equal to constant functions and these identifications combine to produce a path $p : f(0) = f_0$.

If we write $\overline{\text{coe}}_C$ for the “backwards” coercion function induced by $\langle \text{op} \mid \text{isCov}(C) \rangle$, we may describe f symbolically as follows:

$$f = \lambda i c. \text{coe}_{D(i \wedge -)}(f_0(\overline{\text{coe}}_{C(-i \wedge -)}(c)))$$

Suppose now we are given $g : (i : \mathbb{1}) \rightarrow \langle \text{op} \mid C(-i) \rangle \rightarrow D(i)$ along with $p : g(0) = f_0$. We must then show that $(f, p) = (g, q)$. Let us consider the following H :

$$H : (i j : \mathbb{1}) \rightarrow \langle \text{op} \mid C(-i) \rangle \rightarrow D i$$

$$H = \lambda i j. c. \text{coe}_{D((i \wedge j) \vee - \wedge i)}(g(i \wedge j) (\overline{\text{coe}}_{C(-(i \wedge j) \wedge - \vee -i)}(c)))$$

We may construct a function $r : (j : \mathbb{1}) \rightarrow H(0, j) = f_0$ using p and q . Moreover, we may identify $(H(-, 0), r(0))$ with (f, p) and $(H(-, 1), r(1))$ with (g, q) . Finally, since the type $(i : \mathbb{1}) \rightarrow$

$\langle \text{op} \mid C(\neg i) \rangle \rightarrow D(i)$ is a groupoid [RSS20, Lemma 1.26] and groupoids are closed under Σ -types, $\lambda j. (H(-, j), r(j))$ induces the required identification between (f, p) and (g, q) as required. \square

These proofs exhibit proof strategies that are common when working with isACov in TT_{\square} : either reducing to a generic global case where various modalities can be simplified or performing several small modal manipulations and then applying standard and non-modal arguments. They are also very similar to the construction of fibrancy structures in Weaver and Licata [WL20]. The major difference between the proofs given in op. cit. stems from the fact that our constructions take place in a univalent type theory. Consequently, our coercion operators have a simpler type, but we must show that they are unique up to a contractible choice rather than merely having to construct some inhabitant.

6 The directed univalent universe

With our preliminary work on amazing covariance in place, we are now in a position to define our directed univalent universe of groupoids \mathcal{S} and establish its core properties. We begin with the (now short) definition of \mathcal{S} :

Definition 6.1. We define \mathcal{S} to be $\sum_{A:\mathcal{U}} \text{isACov}(A)$.

We note that \mathcal{S} can be fully characterized without reference to ACov as a corollary of Lemma 5.13:¹²

Lemma 6.2. *If $A \vDash_b X \rightarrow \mathcal{U}$ then A factors through \mathcal{S} if and only if it is covariant. In other words, \mathcal{S} is the base of the universal covariant family of simplicial types.*

Corollary 6.3. *$A \vDash_b \mathcal{U}$ factors through \mathcal{S} if and only if it is a groupoid.*

Applying Lemma 5.9, we conclude that $\mathcal{S} \rightarrow \mathcal{U}$ factors through \mathcal{U}_{\square} . In other words, all amazingly covariant families of groupoids are automatically simplicial. Moreover, by Lemma 5.14 along with the closure results from Rijke, Shulman, and Spitters [RSS20], we conclude:

Lemma 6.4. *As a subtype of \mathcal{U} , \mathcal{S} is (1) univalent (2) contains only simplicial types (3) closed under dependent sums, equality, and $i = 1$ (4) closed under the two modalized forms of Π -types indicated by Lemma 5.14.*

Thus, we already have established that \mathcal{S} is a subuniverse of \mathcal{U} spanned by groupoids. What remains is to prove directed univalence i.e., to characterize $\mathbb{1} \rightarrow \mathcal{S}$. To this end, we will first prove two important lemmas for constructing elements of \mathcal{S} . With these in place, we shall show that \mathcal{S} is not only closed under various connectives, but also simplicial, Segal, Rezk, and directed univalent. Our main result can be summed up as follows

Theorem 6.5. *\mathcal{S} is a directed-univalent category.*

Remark 6.6. While it is not helpful for establishing the above theorem, the following observation is helpful for seeing that the construction of \mathcal{S} using cubical spaces must yield the same results as a more standard argument within simplicial spaces and, in fact, must produce the standard ∞ -category of ∞ -groupoids.

Suppose that S' was constructed as the base of the universal covariant fibration among *simplicial* types, we can then argue that S' is also necessarily equivalent to the base of the universal covariant fibration among cubical spaces. First, observe that the universal fibration $S'_{\bullet} \rightarrow S'$ is covariant when viewed as a map of cubical spaces: covariance is equivalent to the map $(S'_{\bullet})^{\mathbb{1}} \rightarrow S'_{\bullet} \times_{S'} (S')^{\mathbb{1}}$ being an equivalence and simplicial spaces are a full subcategory of cubical spaces closed under

¹²Theoretically, every result about \mathcal{S} can be proven using this characterization. We will not endeavor to do so and instead optimize for more readable proofs.

limits and exponentials. Next, we see that this entails the existence of a classifying map $S' \rightarrow \mathcal{S}$ and it is routine to calculate that this map induces an equivalence $\text{hom}(\mathbb{N}^n, S') \simeq \text{hom}(\mathbb{N}^n, \mathcal{S})$ since \mathbb{N}^n is simplicial. Consequently, $\mathcal{S} \simeq S'$. In fact, a nearly identical argument shows that S' must in turn agree with the base of a covariant family universal among covariant families of complete Segal spaces (i.e., ∞ -categories) as well—again assuming such a thing exists.

Consequently, if one assumes the ordinary statement of straightening–unstraightening for the ∞ -category of ∞ -groupoids, then the base of the universal covariant family in cubical spaces must agree with the standard ∞ -category of ∞ -groupoids. While not a satisfactory method of constructing \mathcal{S} , this does show that our construction must yield the expected result.

6.1 The two key lemmas

Before we can prove that \mathcal{S} is directed univalent, we require a better understanding of when two maps $\mathbb{N} \rightarrow \mathcal{S}$ are equivalent. In particular, suppose we are given $f, g : \mathbb{N} \rightarrow \mathcal{S}$. We already know that \mathcal{S} is univalent as a subtype of \mathcal{U}_{\square} and so f and g are equal when there is an equivalence $\alpha : (i : \mathbb{N}) \rightarrow f(i) \rightarrow g(i)$. Accordingly, it suffices to find conditions to establish that $\alpha(i)$ is an equivalence for each $i : \mathbb{N}$. Our first result shows that this holds everywhere if it holds at 0 and 1. In other words, to check that a natural transformation α is an equivalence, it suffices to check that it is an equivalence at each object. We prove a slight generalization of this result which applies to any Δ^ℓ rather than just Δ^1 .

Notation 6.7. We denote $(1, \dots, 1, 0, \dots, 0) : \Delta^\ell$ with k copies of 1 followed by $\ell - k$ copies of 0 by \bar{k} .

Lemma 6.8. *Fix $\ell :_{\mathbb{N}} \text{Nat}$ and suppose that $f, g : \Delta^\ell \rightarrow \mathcal{S}$ and $\alpha : (\delta : \Delta^\ell) \rightarrow f \delta \rightarrow g \delta$ then α is invertible if and only if $\alpha \bar{k} : f \bar{k} \rightarrow g \bar{k}$ is invertible for all $k \leq \ell$.*

PROOF. We begin by generalizing to apply **Axiom 8**. To this end, fix the following global types:

$$\begin{aligned} X &= \sum_{F, G : \Delta^\ell \rightarrow \mathcal{S}} \sum_{\alpha : (\delta : \Delta^\ell) \rightarrow F \delta \rightarrow G \delta} (k : \text{Nat}_{\leq \ell}) \rightarrow \text{isEquiv}(\alpha \bar{k}) \\ Y &= \sum_{F, G : \Delta^\ell \rightarrow \mathcal{S}} \sum_{\alpha : (\delta : \Delta^\ell) \rightarrow F \delta \rightarrow G \delta} \prod_{\delta : \Delta^\ell} \text{isEquiv}(\alpha(\delta)) \end{aligned}$$

It suffices to show that the forgetful map $Y \rightarrow X$ is an equivalence and so, by **Axiom 8**, we must show that for each $n :_{\mathbb{N}} \text{Nat}$ the map $\langle b \mid \mathbb{N}^n \rightarrow Y \rangle \rightarrow \langle b \mid \mathbb{N}^n \rightarrow X \rangle$ is an equivalence. For clarity, we write $\Gamma = \mathbb{N}^n$ and $\Gamma' = \mathbb{N}^n \times \Delta^\ell$ in what follows.

We now unfold this slightly. Fix $F, G :_{\mathbb{N}} \Gamma' \rightarrow \mathcal{S}$ along with $\alpha :_{\mathbb{N}} ((v, \delta) : \Gamma') \rightarrow F(v, \delta) \rightarrow G(v, \delta)$ and $e :_{\mathbb{N}} (v : \Gamma)(k : \text{Nat}_{\leq \ell}) \rightarrow \text{isEquiv}(\alpha(v, \bar{k}))$. We must show the following:

$$\langle b \mid ((v, \delta) : \Gamma') \rightarrow \text{isEquiv}(\alpha(v, \delta)) \rangle$$

We can reorient F, G as global families $\pi_F, \pi_G : \tilde{F}, \tilde{G} \rightarrow \Gamma'$. That both F, G factor through \mathcal{S} implies that both projections are both covariant fibrations and, therefore, orthogonal to the maps $\{0\} \rightarrow \mathbb{N}^m$ for any $m :_{\mathbb{N}} \text{Nat}$. Note, too, that from this viewpoint, α is a map $\tilde{\alpha} : \tilde{F} \rightarrow \tilde{G}$ over Γ' such that pulling back along $(\text{id}, \bar{k}) : \Gamma \rightarrow \Gamma'$ induces an equivalence. We must show that $\tilde{\alpha}$ is an equivalence.

By another application **Axiom 8**, to show that $\tilde{\alpha}$ is an equivalence we must show it induces an equivalence $\langle b \mid \mathbb{N}^m \rightarrow \tilde{F} \rangle \simeq \langle b \mid \mathbb{N}^m \rightarrow \tilde{G} \rangle$. By orthogonality, we note that $\langle b \mid \mathbb{N}^m \rightarrow \tilde{F} \rangle \simeq \langle b \mid \tilde{F} \times_{\Gamma'} (\mathbb{N}^m \rightarrow \Gamma') \rangle$. Consequently, it suffices to show that the following map is an equivalence:

$$\langle b \mid \tilde{F} \times_{\Gamma'} (\mathbb{N}^m \rightarrow \Gamma') \rangle \rightarrow \langle b \mid \tilde{G} \times_{\Gamma'} (\mathbb{N}^m \rightarrow \Gamma') \rangle$$

We may refactor this using the various properties of $\langle b \mid - \rangle$ to obtain the following equivalent map:

$$\sum_{v :_{\mathbb{N}} \mathbb{N}^m \rightarrow \Gamma} \sum_{\theta :_{\mathbb{N}} \mathbb{N}^m \rightarrow \Delta^\ell} \langle b \mid F(v(\vec{0}), \theta(\vec{0})) \rangle \rightarrow \sum_{v :_{\mathbb{N}} \mathbb{N}^m \rightarrow \Gamma} \sum_{\theta :_{\mathbb{N}} \mathbb{N}^m \rightarrow \Delta^\ell} \langle b \mid F(v(\vec{0}), \theta(\vec{0})) \rangle$$

Finally, $\theta(\vec{0})$ is an element of $\langle b \mid \Delta^\ell \rangle$ and is therefore equal to \vec{k} for some k by [Axiom 7](#). For any k , the map is an equivalence as it is derived from α and our conclusion follows. \square

Remark 6.9. Weaver and Licata [[WL20](#)] axiomatized their cobar modality to formulate and postulate a special case of this lemma (their *equivalence axiom*). In our case, no such steps are required as this result follows from [Axiom 8](#). This is not to say that the cobar construction plays no role in our setting: it is used in [Section 4](#) implicitly as Shulman [[Shu19](#)] uses it to characterize the injective fibrations we use to model types.

To ensure that elements of \mathcal{S} are indeed groupoids, $A : \mathcal{U}$ lands in the subtype \mathcal{S} only when it is simplicial in addition to being amazingly covariant. Often, it is easiest to do this by proving that A is amazingly covariant and then applying \boxtimes to A to obtain a simplicial type. In order for this to be possible, however, we must know that applying \boxtimes to an amazingly covariant type results in an amazingly covariant type. The next lemma proves (a generalization of) this fact.

Let us note that the canonical maps $\pi : \mathcal{S} \rightarrow \mathcal{U}$ and $\hat{\boxtimes} : \boxtimes \mathcal{U} \rightarrow \mathcal{U}$ induce a map $\hat{\boxtimes}\pi : \boxtimes \mathcal{S} \rightarrow \mathcal{U}$. Showing that $\boxtimes A$ is amazingly covariant if A is amazingly covariant corresponds to showing that $\hat{\boxtimes}\pi \circ \eta$ factors through \mathcal{S} . We prove this by proving the following stronger result:

Lemma 6.10 (Simplicial exchange). $\hat{\boxtimes}\pi : \boxtimes \mathcal{S} \rightarrow \mathcal{U}$ factors through \mathcal{S} .

PROOF. Given that the composite $\hat{\boxtimes}\pi : \boxtimes \mathcal{S} \rightarrow \mathcal{U}$ mentions no free variables, it suffices by [Lemma 5.13](#) to show that $\hat{\boxtimes}\pi$ is covariant.

For concision, we write $X = \boxtimes \mathcal{S}$ and \tilde{X} for $\sum_{A:X} \hat{\boxtimes} A$. We must show that the map given by evaluating at 0 induces an equivalence between $\mathbb{1} \rightarrow \tilde{X}$ and $(\mathbb{1} \rightarrow X) \times_X \tilde{X}$. Using [Axiom 8](#) along with the observation that these types are all simplicial, it suffices to show that the following map is an equivalence for all $n : \mathbb{N}$:

$$\langle b \mid \Delta^n \times \mathbb{1} \rightarrow \tilde{X} \rangle \rightarrow \langle b \mid \Delta^n \rightarrow (\mathbb{1} \rightarrow X) \times_X \tilde{X} \rangle$$

In other words, we must show that $\Delta^n \times \{0\} \rightarrow \Delta^n \times \mathbb{1}$ is *globally* orthogonal to $\tilde{X} \rightarrow X$.

We now argue that to prove this, it suffices to show that $\tilde{X} \rightarrow X$ is globally orthogonal to $\{0\} \rightarrow \Delta^k$ for all k . Let us assume that this condition holds for the moment and show that it suffices to establish orthogonality with respect to $\Delta^n \times \{0\} \rightarrow \Delta^n \times \mathbb{1}$. Considering the sequence of maps $\{0\} \rightarrow \Delta^n \times \{0\} \rightarrow \Delta^n \times \mathbb{1}$, a standard 3-for-2 argument shows that it suffices to show that $\{0\} \rightarrow \Delta^n \times \mathbb{1}$ is orthogonal to $\tilde{X} \rightarrow X$.

To this end, notice that $\Delta^n \times \mathbb{1}$ is a subtype of \mathbb{N}^{n+1} spanned by tuples (v_1, \dots, v_n, w) where $v_1 \geq \dots \geq v_n$. Observing that both \tilde{X} and X are simplicial, it follows that a map $\Delta^n \times \mathbb{1} \rightarrow X$ or $\Delta^n \times \mathbb{1} \rightarrow \tilde{X}$ is the same as a map from the following subtype of \mathbb{N}^{n+1} :

$$\{(v_1, \dots, v_n, w) \mid (w \geq v_1 \geq \dots \geq v_n) \vee (v_1 \geq w \geq \dots \geq v_n) \vee \dots \vee (v_1 \geq \dots \geq v_n \geq w)\}$$

Let us denote the condition $v_1 \geq \dots \geq v_k \geq w \geq v_{k+1} \geq \dots \geq v_n$ by $\Phi_{n,k}$ with $\Phi_{n,0}$ being $w \geq v_1 \geq \dots \geq v_n$. We wish to show that for any element $\tilde{x} : \tilde{X}$ and map $f : \Delta^n \times \mathbb{1} \rightarrow X$ along with a path $p : \pi \tilde{x} = f(0, \dots, 0)$ that there exists a unique extension of f to \tilde{f} . We may phrase this as constructing an element of the following type:

$$\text{isContr}((t : \Delta^n \times \mathbb{1}) \rightarrow \sum_{\tilde{y}:\tilde{X}} \sum_{q:\pi(\tilde{y})=f(t)} t = 0 \rightarrow \sum_{r:\tilde{x}=\tilde{y}} p = \pi(r) \bullet q)$$

Since we are therefore constructing a map out of a series of disjunctions (and constructing an identification of such maps) it suffices to show that this proposition is inhabited for each subtype $\{(v_1, \dots, v_n, w) \mid \Phi_{n,k_0} \wedge \dots \wedge \Phi_{n,k_i}\}$ for each sequence i and $k_0 \leq k_1 \leq \dots \leq k_i$. Calculation reveals that the intersection $\Phi_{n,k_0} \wedge \dots \wedge \Phi_{n,k_i}$ is necessarily a sub-simplex of $\Delta^n \times \mathbb{1}$. Consequently, unfolding

definitions, constructing such a map on this subtype is precisely the same as showing that $\tilde{X} \rightarrow X$ is globally orthogonal $\{0\} \rightarrow \Delta^k$ for some k , our assumption.

All told then, it suffices to show the following canonical map is an equivalence:

$$\langle b \mid \Delta^m \rightarrow \tilde{X} \rangle \rightarrow \langle b \mid (\Delta^m \rightarrow X) \times_X \tilde{X} \rangle$$

By [Axiom 9](#), we may “remove the \square ” from X and \tilde{X} and so this type is equivalent to the following:

$$\langle b \mid \Delta^m \rightarrow \sum_{A:\mathcal{S}} A \rangle \rightarrow \langle b \mid (\Delta^m \rightarrow \mathcal{S}) \times_{\mathcal{S}} (\sum_{A:\mathcal{S}} A) \rangle$$

This, finally, is an equivalence because $(\sum_{A:\mathcal{S}} A) \rightarrow \mathcal{S}$ is covariant ([Lemma 5.11](#)). \square

Corollary 6.11. *\mathcal{S} is closed under coequalizers in \mathcal{U}_{\square} .*

PROOF. By [Lemma 5.14](#), $\mathcal{U}_{\text{ACov}}$ is closed under coequalizers $\text{Coeq}(f, g)$ and so [Lemma 6.10](#) ensures the $\square \text{Coeq}(f, g)$ lands in \mathcal{S} as well. By Rijke, Shulman, and Spitters [[RSS20](#)], this is the coequalizer in \mathcal{U}_{\square} . \square

6.2 \mathcal{S} is directed univalent, Segal, Rezk, and simplicial

We are now able to show that \mathcal{S} satisfies all the desired properties for a universe of groupoids. We begin by showing that we have, at last, constructed a directed univalent universe.

First, we note that [Definition 1.2](#) merely states that there is some isomorphism between two types. We are already in a position to construct one of these two maps:

Lemma 6.12. *There is a function mor2fun from $\mathbb{1} \rightarrow \mathcal{S}$ to $\sum_{A,B:\mathcal{S}} A \rightarrow B$.*

PROOF. Given $F : \mathbb{1} \rightarrow \mathcal{S}$, by [Lemma 5.11](#) this induces a covariant family $F_0 : \mathbb{1} \rightarrow \mathcal{U}$. We then define $\text{mor2fun}(F) := (F_0 \ 0, F_0 \ 1, \text{coe}_F)$ where the last component is induced by [Lemma 5.5](#). \square

Theorem 6.13 (Directed univalence). *The function mor2fun is an equivalence.*

Prior to proving this result, we will construct a putative inverse to mor2fun .

Definition 6.14. Given $A, B : \mathcal{S}$ and $f : A \rightarrow B$, $\text{Gl}(A, B, f) : \mathbb{1} \rightarrow \mathcal{S}$ is $\lambda i. \sum_{b:B} i = 0 \rightarrow f^{-1}(b)$.

Gl is the directed version of the glue type from cubical type theory [[Coh+17](#); [Sat17](#)] and is inspired directly from the construction used in Weaver and Licata [[WL20](#)] in their construction of a directed univalent universe.¹³ In our case, we have no need to add it as a primitive in our setting: this was necessary in (bi)cubical type theory to achieve certain definitional equalities, but we are pervasively working up to equivalence. We note that $\text{Gl}(f)$ factors through \mathcal{S} by virtue of (2–4,6) of [Lemma 5.14](#) along [Axiom 2](#) which ensures that $\langle \text{op} \mid \neg j = 1 \rangle = (j = 0)$. We also record a few elementary fact about Gl below:

Lemma 6.15. *Given A, B, f as above, $\text{Gl}(A, B, f) \ 0 = A$, $\text{Gl}(A, B, f) \ 1 = B$, and $\text{coe}_{\text{Gl}(A,B,f)} = f$.*

We now return to the proof of [Theorem 6.13](#).

PROOF OF [THEOREM 6.13](#). We will prove that Gl forms a quasi-inverse to mor2fun and thereby conclude that mor2fun is an equivalence. We must therefore prove (1) $\text{mor2fun} \circ \text{Gl} = \text{id}$ and (2) $\text{Gl} \circ \text{mor2fun} = \text{id}$. (1) follows from direct calculation and [Lemma 6.15](#), so we will detail only (2).

Suppose we are given $F : \mathbb{1} \rightarrow \mathcal{S}$. We must show that $F = \text{Gl}(\text{mor2fun}(F))$ or equivalently, using the fact that \mathcal{S} is univalent, that there is an equivalence $\alpha : (i : \mathbb{1}) \rightarrow F(i) \simeq \text{Gl}(\text{mor2fun}(F)) \ i$. To prove this, we will begin by constructing α and then use [Lemma 6.8](#) to reduce to checking that α is

¹³Not to be confused with *Artin gluing* from categorical logic.

an equivalence at 0 and 1. It is helpful to do this in stages and so we begin by supposing $i : \mathbb{1}$ and $z : F(i)$ and define α as follows for some X and Y to be determined:

$$\alpha i z = (X : F(1), Y : i = 0 \rightarrow \text{coe}_F^{-1}(X))$$

We will construct X and Y separately.

We can substantiate X immediately: $\text{coe}_{F(-\vee i)} : F(i) \rightarrow F(1)$ and so we choose $X := F(-\vee i) z$. This refines the type of Y to $i = 0 \rightarrow \text{coe}_F^{-1}(\text{coe}_{F(-\vee i)} z)$. Assume $\phi : i = 0$ so that it suffices to define $Y.1 : F(0)$ and $Y.2 : \text{coe}_F Y.1 = \text{coe}_{F(-\vee i)} z$. Using ϕ , we may suppose that $z : F(0)$ and that the type of $Y.2$ is $\text{coe}_F Y.1 = \text{coe}_F z$ (since $0 \vee - = \text{id}$). After this, $Y.1 := z$ and $Y.2 := \text{refl}$ suffices.

Finally, it is now straightforward to check that $\alpha 0$ and $\alpha 1$ are equivalences using [Lemma 6.15](#). \square

The proof that \mathcal{S} is Segal is very similar to the proof of directed univalence, though not quite a consequence of it. Since the proof is similar to [Theorem 6.13](#), we provide only a sketch.

Lemma 6.16. *\mathcal{S} is Segal.*

PROOF SKETCH. We must show that $(\Delta^2 \rightarrow \mathcal{S}) \rightarrow (\Delta_1^2 \rightarrow \mathcal{S})$ is an equivalence. We begin by noting that the codomain can be rewritten with [Theorem 6.13](#) as $T = \sum_{A B C : \mathcal{S}} A \rightarrow B \times B \rightarrow C$. We only need to show that the forgetful map from $(\Delta^2 \rightarrow \mathcal{S}) \rightarrow T$ is an equivalence.

This argument proceeds along the same lines as [Theorem 6.13](#) where we replace $\mathbb{1}$ with Δ^2 : we introduce a variant of Gl which glues together three spaces along two maps and show that this procedure induces a quasi-inverse to the forgetful map $(\Delta^2 \rightarrow \mathcal{S}) \rightarrow T$. It is here that we require [Lemma 6.8](#) with $\ell = 2$ rather than $\ell = 1$. \square

Corollary 6.17. *Composition of the morphisms in \mathcal{S} is realized by ordinary function composition.*

In particular, an invertible morphism corresponds via [Theorem 6.13](#) to an equivalence. Combining this with ordinary univalence, we obtain:

Corollary 6.18. *\mathcal{S} is Rezk.*

Our final result is that \mathcal{S} lands in the subuniverse of simplicial types.

Lemma 6.19. *\mathcal{S} is simplicial.*

PROOF. By Rijke, Shulman, and Spitters [[RSS20](#), Lemma 1.20], it suffices to show that $\eta : \mathcal{S} \rightarrow \square \mathcal{S}$ has a retraction. By univalence, the composite of $\eta : \mathcal{U}_{\square} \rightarrow \square \mathcal{U}_{\square}$ followed by $\hat{\square} : \square \mathcal{U}_{\square} \rightarrow \mathcal{U}_{\square}$ is the identity and so it suffices to show that both these maps restrict to \mathcal{S} . That is, it suffices to show that $\hat{\square} \circ \pi : \square \mathcal{S} \rightarrow \mathcal{U}$ factors through \mathcal{S} . This is an immediate consequence of [Lemma 6.10](#). \square

We conclude by noting a few of the categorical properties \mathcal{S} enjoys:

Lemma 6.20. *\mathcal{S} is finitely complete and finitely cocomplete and satisfies descent [[Rij18](#), Chapter 2].*

PROOF SKETCH. Finite completeness and cocompleteness are an immediate consequence of [Lemma 6.4](#) and [Corollary 6.11](#) along with [Theorem 6.13](#) which implies that a e.g., categorical limit in \mathcal{S} is an ordinary HoTT limit of groupoids. To prove the descent properties, we must show that various limits and colimits commute appropriately. However, by [Theorem 6.13](#) once more, this is an immediate consequence of the fact that limits and colimits in HoTT enjoy descent [[Rij18](#)]. \square

7 Consequences of a directed univalent universe

We now reap the rewards of our efforts in constructing \mathcal{S} and give a brief tour of the consequences of this type. We show how directed univalence may be used to prove free theorems and substantiate the structure homomorphism principle. We also use it to construct various foundational example categories and lay the groundwork for the development of *higher algebra* within TT_{\square} .

7.1 Free theorems from naturality

Directed univalence allows us to make a precise link between familiar parametricity arguments [Wad89] with the categorical naturality arguments that helped motivate them. In particular, directed univalence implies that a function $\alpha : (A : \mathcal{S}) \rightarrow F(A) \rightarrow G(A)$ is natural:

Lemma 7.1. *If $F_0, F_1 : \mathcal{S} \rightarrow \mathcal{S}$ and $\alpha : (A : \mathcal{S}) \rightarrow F_0(A) \rightarrow F_1(A)$ then $\alpha(B) \circ F_0(f) = F_1(f) \circ \alpha(A)$ for any $f : A \rightarrow B$.*

PROOF. Fix $A, B : \mathcal{S}$ along with $f : A \rightarrow B$ and denote the corresponding morphism $G : \mathbb{1} \rightarrow \mathcal{S}$. Note that $\alpha \circ G$ is then a function $(i : \mathbb{1}) \rightarrow F_0(i) \rightarrow F_1(i)$. Applying [Theorem 6.13](#) once more, we note that $\alpha(G(i)) : F_0(i) \rightarrow F_1(i)$ is a morphism in \mathcal{S} for every i . Accordingly, $\alpha \circ G$ is equivalent to some $s : (i j : \mathbb{1}) \rightarrow H i j$ for some H where $H i 0 = F_0 i$ and $H i 1 = F_1 i$. We visualize H as:

$$\begin{array}{ccc} F_0 0 & \xrightarrow{F_0} & F_0 1 \\ \alpha(G 0) \downarrow & & \downarrow \alpha(G 1) \\ F_1 0 & \xrightarrow{F_1} & F_1 1 \end{array}$$

This commuting square is equivalently an equality between the composites F_1 and $\alpha(G 0)$ and $\alpha(G 1)$ and F_0 . The conclusion then follows from [Corollary 6.17](#). \square

Lemma 7.2. *If $f : (A : \mathcal{S}) \rightarrow A \rightarrow A$ then $f = \lambda A a. a$.*

PROOF. Fix $A : \mathcal{S}$ and suppose we are given $a : A$. Applying [Lemma 7.1](#) to f and $\lambda_. a$, we conclude that $f A (a \star) = a(f \mathbf{1} \star)$. Since $f \mathbf{1} \star = \star$ by the η principle of $\mathbf{1}$, $f = \lambda A a. a$. \square

Nothing limits us to considering only operations $\mathcal{S} \rightarrow \mathcal{S}$. The same techniques scale to multi-argument operations such as $\mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$ or even mixed-variance operations such as $\langle \text{op} \mid \mathcal{S} \rangle \times \mathcal{S} \rightarrow \mathcal{S}$:

Lemma 7.3. *If $\alpha : (A B : \mathcal{S}) \rightarrow A \times B \rightarrow A$ then $\alpha = \pi_1$.*

Lemma 7.4. *If $A, B :_{\flat} \mathcal{S}$ and $\alpha : (C :_{\text{op}} \mathcal{S}) \rightarrow A^{\langle \text{op} \mid C \rangle} \rightarrow B^{\langle \text{op} \mid C \rangle}$ then $\alpha = \lambda_g. f \circ g$ for some $f : A \rightarrow B$.*

This methodology highlights the *limitations* of naturality as a facsimile for parametricity: for operations whose parameters are not used strictly co- or contravariantly, directed univalence does not provide any free theorems. We leave it to future work to consider alternative universes of correspondences [AF20] and what parametricity arguments they might provide.

7.2 Full subcategories of \mathcal{S}

A large number of important categories can be described as a *full subcategories* of \mathcal{S} . To do this, we must first show how to obtain full subcategories inside of $\mathbb{T}\mathbb{T}_{\square}$. Recall that a full subcategory of C_0 of a category $C :_{\flat} \mathcal{U}$ is a category C_0 where objects are a subset of those in C but the morphisms and all the higher cells agree. In other words, a full subcategory is described by a predicate $(b \mid C) \rightarrow \text{HProp}_{\square}$ which picks out those objects which land in C_0 .

Definition 7.5. Given $\phi :_{\sharp} (b \mid C) \rightarrow \text{HProp}_{\square}$, the resulting full subcategory C_{ϕ} is $\sum_{c:C} (\# \mid \phi(c^{\eta}))$.¹⁴

¹⁴In practice, ϕ will be b -annotated.

Here we for the first time have occasion to explicitly use the right adjoint \sharp to \flat . Let us note that C_ϕ is a category because (1) categories are closed under dependent sums and (2) $\langle \sharp \mid \phi(c^\eta) \rangle$ is a groupoid. Furthermore, we can prove that C_ϕ is actually a full subcategory:

Lemma 7.6. *Given C and ϕ as above, if $a, b : C_\phi$ then $\text{hom}_{C_\phi}(a, b) \simeq \text{hom}_C(\pi_1 a, \pi_1 b)$.*

PROOF. Unfolding definitions, it suffices to show the following two propositions are equivalent for all $f : \mathbb{1} \rightarrow C$:

$$\begin{aligned} &\langle \sharp \mid \phi(f^\eta(0)) \rangle \times \langle \sharp \mid \phi(f^\eta(1)) \rangle \\ &(i : \mathbb{1}) \rightarrow \langle \sharp \mid \phi(f^\eta(i^\eta)) \rangle \end{aligned}$$

Since $\flat \dashv \sharp$, we may use [Lemma 3.14](#) to replace the second proposition with $\langle \sharp \mid (i \flat \mathbb{1}) \rightarrow \phi(f^\eta(i)) \rangle$. Finally, [Axiom 7](#) tells us that $\langle \flat \mid \mathbb{1} \rangle$ is equivalent to Bool and we may replace $(i \flat \mathbb{1}) \rightarrow \phi(f^\eta(i))$ with $(b : \text{Bool}) \rightarrow \phi(f^\eta(\text{if } b \text{ then } 0 \text{ else } 1))$ and conclusion follows. \square

Lemma 7.7. *Given a category $C \flat \mathcal{U}$ and $\phi \flat (b \mid C) \rightarrow \text{HProp}_\square$ then $a \flat C$ is an element of C_ϕ if and only if $\phi(a)$ holds.*

By choosing different predicates on \mathcal{S} we obtain a number of familiar categories. For instance:

Definition 7.8. The category of n -truncated groupoids $\mathcal{S}_{\leq n}$ is given by $\mathcal{S}_{\text{hasHLevel}(n+2)}$.¹⁵ In particular, the category of *propositions* is given by $\mathcal{S}_{\leq -1}$, and the category of *sets* is given by $\mathcal{S}_{\leq 0}$.

Definition 7.9. The category of finite sets \mathcal{F} is given by \mathcal{S}_ϕ where $\phi(X) = \sum_{n:\text{Nat}} (X = \text{Nat}_{\leq n})$.

Note that \mathcal{F} is quite different than $\sum_{A:\mathcal{S}} \exists n. \text{Nat}_{\leq n} = A$, which has only invertible morphisms. The definition of C_ϕ is necessary to ensure that ϕ is applied only to the objects of C , not its higher cells.

[Lemma 7.6](#) implies that these examples inherit directed univalence from \mathcal{S} , the first instance of the *structure homomorphism principle (SHP)* [[WL20](#)]: homomorphisms in structured types coincide with their standard analytic formulations and, consequently, all terms and types are functorial for these analytic morphisms. For instance, a morphism in \mathcal{F} corresponds to an ordinary function and, consequently, a family $F : \mathcal{F} \rightarrow \mathcal{S}$ has an action $F(A) \rightarrow F(B)$ for any ordinary function $A \rightarrow B$.

7.3 The structure homomorphism principle

Not only full subcategories of \mathcal{S} enjoy SHP, in this section we survey other categories which satisfy it as well. As a prototypical example, we consider pointed spaces, $\mathcal{S}_* = \sum_{A:\mathcal{S}} A$:

Lemma 7.10. *Homomorphisms $\text{hom}_{\mathcal{S}_*}((A, a), (B, b))$ are equal to pointed functions $\sum_{f:A \rightarrow B} f(a) = b$.*

PROOF. By [Lemma 6.2](#), the projection map $\mathcal{S}_* \rightarrow \mathcal{S}$ is covariant, giving, for any pair of pointed spaces (A, a_0) and (B, b_0) , an equivalence between homomorphisms from a_0 to b_0 lying over a homomorphism $f : A \rightarrow B$ and identifications $f(a_0) = b_0$. \square

This same methodology can be applied to more general algebraic structures to yield categories of e.g., monoids, groups, rings, etc. which we conjecture all enjoy SHP. Rather than dealing with this generality, we will focus on monoids to complete the example given in [Section 1](#). We recall the type of monoids:

$$\text{Monoid} = \sum_{A:\mathcal{S}_{\leq 0}} \sum_{\epsilon:A} \sum_{\cdot:A \times A \rightarrow A} \text{isAssociative}(\cdot) \times \text{isUnit}(\cdot, \epsilon)$$

By repeated application of the closure of categories under dependent sums, functions, and equalities, we already conclude that Monoid is a category. Moreover, we can characterize its homomorphisms.

¹⁵The correction +2 ensures that $\mathcal{S}_{\leq n}$ comports with the standard indexing in homotopy theory which begins at -2 , not 0 .

Lemma 7.11. *A homomorphism $\text{hom}((A, \epsilon_A, \cdot_A, \alpha_A, \mu_A), (B, \epsilon_B, \cdot_B, \alpha_B, \mu_B))$ is precisely a standard monoid homomorphism e.g. a function $A \rightarrow B$ commuting with multiplication and the unit.*

PROOF. To show that the type `Monoid` is a category, we proceed in several steps. Let us denote by $\mathcal{S}_{\leq 0, *}$ the category of pointed sets $\sum_{A: \mathcal{S}_{\leq 0}} A$. We obtain define the type $\text{Mag} := \sum_{A: \mathcal{S}_{\leq 0, *}} A \times A \rightarrow A$ of *magmas* via the pullback, showing that it is a category:

$$\begin{array}{ccc} \text{Mag} & \longrightarrow & \mathcal{S}_{\leq 0}^{\parallel} \\ \downarrow & \lrcorner & \downarrow (s, t) \\ \text{Unit} & \xrightarrow{(A \times A, A)} & \mathcal{S}_{\geq 0} \times \mathcal{S}_{\geq 0} \end{array}$$

We get a natural projection from the type Mag_* of *pointed magmas* which is also a covariant fibration as evinced by the following diagram:

$$\begin{array}{ccccc} \text{Mag}_* & \longrightarrow & \mathcal{S}_{\leq 0, *} & \xrightarrow{\quad} & \mathcal{S}_* \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ \text{Mag} & \longrightarrow & \mathcal{S}_{\geq 0} & \xrightarrow{\quad} & \mathcal{S} \end{array}$$

The type `Monoid` is then a subtype of Mag_* , so $\text{Monoid} \simeq \text{Mag} \times_{\text{prop}} \text{Unit}$, and hence `Monoid` is a category.

We compute the (free) morphisms in `Monoid` as follows:

$$\begin{aligned} \mathbb{1} &\rightarrow \text{Monoid} \simeq \mathbb{1} \rightarrow \sum_{A: \mathcal{S}_{\leq 0}} \sum_{\epsilon_A: A} \sum_{\mu_A: A \times A \rightarrow A} \text{isAssociative}(\mu_A) \times \text{isUnit}(\mu_A, \epsilon_A) \\ &\simeq (E: \mathbb{1} \rightarrow \mathcal{S}_{\leq 0}) \times (\epsilon: \prod_{i: \mathbb{1}} E(i)) \times (\mu: \prod_{i: \mathbb{1}} E(i) \times E(i) \rightarrow E(i)) \\ &\quad \times (\text{isAssociative}(\mu(i))) \times \text{isUnit}(\mu(i), \epsilon(i)) \\ &\stackrel{\text{Theorem 6.13}}{\simeq} (E: \mathbb{1} \rightarrow \mathcal{S}_{\leq 0}) \times (\epsilon: \prod_{i: \mathbb{1}} E(i)) \times (\mu: \prod_{i: \mathbb{1}} \text{hom}_{\mathcal{S}_{\leq 0}}(E(i) \times E(i), E(i))) \\ &\quad \times (\text{isAssociative}(\mu(i))) \times \text{isUnit}(\mu(i), \epsilon(i)) \\ &\simeq (E: \mathbb{1} \rightarrow \mathcal{S}_{\leq 0}) \times (\epsilon: \prod_{i: \mathbb{1}} E(i)) \times (\mu: \prod_{i, j: \mathbb{1} \times \mathbb{1} \rightarrow \mathcal{S}_{\leq 0}} \text{hom}_{\mathcal{S}_{\leq 0}}(E(i) \times E(j), E(i))) \\ &\quad \times \prod_{i: \mathbb{1}} (\text{isAssociative}(\lambda j. \mu(i, j))) \times \text{isUnit}(\lambda j. \mu(i, j), \epsilon(i)) \\ &\stackrel{\text{Theorem 6.13}}{\simeq} \sum_{\substack{A, B: \mathcal{S}_{\leq 0} \\ f: A \rightarrow B}} \sum_{\substack{\epsilon_A: A \\ \epsilon_B: B}} \sum_{\substack{\mu_A: A \times A \rightarrow A \\ \mu_B: B \times B \rightarrow B}} (f(a) = b) \times \sum_{\mu_B: B \times B \rightarrow B} f \circ \mu_A = \mu_B \circ (f \times f) \\ &\quad \times \text{isAssociative}(\mu_A) \times \text{isAssociative}(\mu_B) \times \text{isUnit}(\epsilon_A) \times \text{isUnit}(\epsilon_B) \end{aligned}$$

In the final step, we have additionally used the characterization of squares $\mathbb{1} \times \mathbb{1} \rightarrow A$ in a category A as (homotopy) commutative squares of morphisms. For illustration, we can represent the square μ in terms of the family E as follows:

$$\begin{array}{ccc}
A & \xrightarrow{E} & B \\
\uparrow \mu(0, -) & & \uparrow \mu(1, -) \\
A \times A & \xrightarrow{E \times E} & B \times B
\end{array}$$

Overall, the claim follows by taking fibers of $\text{Monoid}^{\flat} \rightarrow \text{Monoid} \times \text{Monoid}$, using the above characterization. \square

Substituting [Lemma 7.11](#) within [Lemma 7.1](#), we obtain the promised result:

Lemma 7.12. *If $F, G : \text{Monoid} \rightarrow \mathcal{S}$ and $\alpha : (A : \text{Monoid}) \rightarrow F(A) \rightarrow G(A)$ then α is natural i.e. if $f : A \rightarrow B$ is a monoid homomorphism, then $\alpha(B) \circ F(f) = G(f) \circ \alpha(A)$.*

To complete our goal of proving sum natural automatically, it remains only to define List as an endomap of monoids where $\text{List } A$ has pointwise multiplication. Remarkably, this is straightforward consequence of our results. One need only write down the definition of this monoid in the ordinary way and conclude that it lifts to a functor because the carrier ($\text{List} = \sum_{n:\text{Nat}} -^n$) is already known to be a functor $\mathcal{S}_{\leq 0} \rightarrow \mathcal{S}_{\leq 0}$ using the closure under Σ and Nat ; no special argument is required.

We can also apply directed univalence to non-algebraic structures using our ability to define n -presheaf categories $\text{PSh}_n(C) = \langle \text{op} \mid C \rangle \rightarrow \mathcal{S}_{\leq n}$. We consider the representative example of partial orders, which we isolate as a full subcategory of a presheaf category. In particular, we begin with the category of reflexive graphs: $\text{RGraph} = \text{PSh}_0(\Delta_{\leq 1})$ where $\Delta_{\leq 1}$ is the “walking fork” given by the pushout $\Delta^2 \sqcup_{\parallel} \Delta^2$ adjoining a pair of retractions ∂_0, ∂_1 to a single arrow $r : 1 \rightarrow 0$. While we have not ensured $\Delta_{\leq 1}$ is a category, this does not matter as RGraph is a category regardless.

We use directed univalence to characterize this category’s *objects* as well as its higher structure:

Lemma 7.13. *The category RGraph is equivalent to $\sum_{G_0:\mathcal{S}_{\leq 0}} \sum_{G_1:G_0 \times G_0 \rightarrow \mathcal{S}_{\leq 0}} \prod_{x:G_0} G_1(x, x)$.*

PROOF. Using the universal property of a pushout, $\text{RGraph} = \mathcal{S}_{\leq 0}^{\Delta^2} \times_{\mathcal{S}^{\Delta^2}} \mathcal{S}_{\leq 0}^{\Delta^2}$ and so repeated application of [Theorem 6.13](#) and [Lemma 6.16](#) proves $\text{RGraph} = \sum_{G_0, G_1:\mathcal{S}_{\leq 0}} \sum_{s:t:G_1 \rightarrow G_0} \sum_{r:G_0 \rightarrow G_1} sr = st$ and the conclusion now follows from a standard argument. \square

We isolate $\text{Pos} \subseteq \text{RGraph}$ as a full subcategory spanned by objects where G_1 is a partial order:

Definition 7.14. $\text{Pos} = \text{RGraph}_{\phi}$ where $\phi(G) := \text{isASym}(G_1) \times \text{isTrans}(G_1) \times \prod_{x,y:G_0} \text{isHProp}(G_1(x, y))$

[Lemma 7.6](#) now proves that homomorphisms in Pos are precisely monotone maps:

Lemma 7.15. *If $P, Q : \text{Pos}$ then $\text{hom}_{\text{Pos}}(P, Q) \simeq \sum_{f:P_0 \rightarrow Q_0} \prod_{x,y:P_0} P_1(x, y) \rightarrow Q_1(fx, fy)$.*

Finally, for the next subsection we isolate a category which is foundational to ∞ -category theory: the simplex category Δ is the full subcategory Pos_{ϕ} where $\phi(P) = \sum_{n:\text{Nat}} P = \Delta^n$.

7.4 First steps in synthetic higher algebra

As homotopy (type) theorists like to quip: homotopy types are modern sets. Higher algebra seeks to take this slogan a step further by studying groups, rings, modules, etc. in a world where homotopy types have replaced sets. While higher algebra has numerous applications to algebraic topology, algebraic K-theory, and algebraic geometry, it is also a notoriously technical: even the simplest higher algebraic structure must account for an infinite tower of coherences for each imposed

equation. For our final application of \mathcal{S} , we initiate the study of *higher algebra* [Lur17; Gep20] in $\mathbb{T}\mathbb{T}_{\square}$ by defining some of the central objects of study. We begin by defining the category of (homotopy-coherent and untruncated) monoids following Segal [Seg74].

Definition 7.16. The *category of coherent monoids* Monoid_{∞} as the full subcategory of $\text{PSh}(\Delta)$ carved out by the following predicate (the Segal condition):

$$\phi(X \text{ ; } \flat \langle \text{op} \mid \Delta \rangle \rightarrow \mathcal{S}) = \text{isContr}((X(\Delta^0))) \times \prod_{n:\text{Nat}_{\geq 1}} \text{isEquiv}(\langle X(\iota_k)_{k < n} \rangle : X(\Delta^n) \rightarrow X(\Delta^1)^n)$$

In the above, $\iota_k : \Delta^1 \rightarrow \Delta^n$ is $\lambda i. (1, \dots, 1, i, 0, \dots)$ picking out k copies of 1.

In other words, a coherent monoid is a functor $X : \langle \text{op} \mid \Delta \rangle \rightarrow \mathcal{S}$ such that $X(\Delta^n)$ is the n -fold product of $X(\Delta^1)$. While somewhat indirect, these conditions encode all the necessary structure e.g., the *unit* is given by the center of contraction $\varepsilon_X : X(\Delta^0)$, and *multiplication* is given by the composite map $\mu_X : X(\Delta^1)^2 \simeq X(\Delta^2) \rightarrow X(\Delta^1)$.

As a small example of manipulating this definition, we prove the following:

Lemma 7.17. *The functor $\text{Monoid}_{\infty} \rightarrow \mathcal{S}$ induced by evaluation at Δ^1 reflects isomorphisms.*

PROOF. Given $f : X \rightarrow Y$, by Riehl and Shulman [RS17] and Lemma 7.6, it suffices to show that if $f(\Delta^1)$ is an isomorphism so is $f(\Delta^n)$ for any n . By the Segal condition and naturality, $f(\Delta^n)$ is equivalent to $(f(\Delta^1))_{i \leq n}$ which is invertible if $f(\Delta^1)$ is an isomorphism. \square

Once again, directed univalence yields that the morphisms in Monoid_{∞} preserve the relevant structure.

Proposition 7.18. *Let $X, Y \text{ ; } \flat \text{ Monoid}_{\infty}$. Then $\text{hom}_{\text{Monoid}_{\infty}}(X, Y)$ is equivalent to the type of natural transformations $X \rightarrow Y$. In particular, for $F \text{ ; } \flat \text{ hom}_{\text{Monoid}_{\infty}}(X, Y)$ we have homotopies $F(\Delta^0)(\varepsilon_X) = \varepsilon_Y$ in $Y(\Delta^0)$, and in the category of spaces \mathcal{S} the following homotopy-commutative diagram:*

$$\begin{array}{ccc} X(\Delta^1)^2 \simeq X(\Delta^2) & \xrightarrow{F(\Delta^2)} & Y(\Delta^2) \simeq Y(\Delta^1)^2 \\ \mu_X \downarrow & & \downarrow \mu_Y \\ X(\Delta^1) & \xrightarrow{F(\Delta^1)} & Y(\Delta^1) \end{array}$$

PROOF. Analogously to [RS17, Proposition 6.6], F is a natural transformation. Then preservation of the unit follows by contractibility, and preservation of multiplication follows by directed univalence and naturality. \square

We can also define the category of coherent groups:

Definition 7.19. The *category of coherent groups* Grp_{∞} is the full subcategory of Monoid_{∞} carved out by the predicate $\phi(X \text{ ; } \flat \text{ Monoid}_{\infty}) = \text{isEquiv}(\lambda x y. (x, \mu(x, y)) : X(\Delta^1)^2 \rightarrow X(\Delta^1)^2)$.

These concepts and many others can be unified through the formalism of (∞) -operads but we leave it to future work to develop this apparatus in $\mathbb{T}\mathbb{T}_{\square}$. An application of such a formalism would be the ability to develop higher algebra not just in \mathcal{S} , but in *spectra*, another fundamental category in modern homotopy theory. We conclude this section by constructing this category.

Suppose C is a pointed category with pullbacks, i.e., C has pullbacks and comes with an element $0 : C$ which is simultaneously initial terminal and initial. Within C , we define the *loop functor* $\Omega : C \rightarrow C$ by $\Omega := \lambda x.0 \times_x 0$. We have already encountered such a pointed category: \mathcal{S}_* .

Definition 7.20. The *category of spectra* Sp is defined as $\lim_{n:\text{Nat}} (\dots \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_* \xrightarrow{\Omega} \mathcal{S}_*)$.

Here \lim refers to the ordinary definition of a limit from HoTT and we note that as the limit of categories, Sp is itself automatically a category. Using directed univalence, we can easily show that objects of Sp are infinite deloopings of a groupoid as expected [Shu13; vDoo18].

8 Conclusions and related work

We have introduced TT_{\square} , an enhancement of simplicial type theory featuring modalities and a relaxed interval type. We have used TT_{\square} as a framework to construct a directed univalent universe of groupoids \mathcal{S} which we have further proven to be a well-behaved category. Finally, we have used \mathcal{S} as a jumping off point to construct numerous examples of categories and categorical reasoning in TT_{\square} relevant both to ∞ -category theory and mechanized verification. In order to do so, we have shown how our same modal operators can be used to e.g., construct full subcategories.

8.1 Related work

While directed type theory generally and simplicial type theory specifically are relatively new areas, there is already substantial work exploring the impact of a “type theory where types are categories.” Much of this work focuses on either constructing such type theories [LH11; War13; Nuy15; RS17; Nor18; Kav19; Nuy20; ANvdW23; NA24; Neu25] or studying “formal” category theory within them [RS17; Wei24c; Wei24b; BW23; Mar24; Mar25] i.e., statements which do not use particular closed non-trivial categories but instead quantify over arbitrary categories. This is distinct from our focus, which has been to combine essentially off-the-shelf type theories [RS17; Gra+21] and to use this combination to prove facts about the concrete type \mathcal{S} and types derived thereof. Closely related to this is the work by Cavallo, Riehl, and Sattler [Rie18] and Weaver and Licata [WL20], who both study directed univalence, in respectively simplicial and *bicubical type theory* (BCTT).

Alternative constructions of \mathcal{S} . Cavallo, Riehl, and Sattler give an alternative construction of \mathcal{S} in the intended model of STT, similar to the classical proof due to Cisinski [Cis19]. They have argued externally that this subuniverse satisfies directed univalence and a version of Lemma 6.2. However, their work is strictly external and does not consider how one might integrate \mathcal{S} within STT. Given that both our universe and theirs satisfy Lemma 6.2, they are weakly equivalent and so our results further show that their universe is e.g., a finitely (co)complete category and closed under various connectives.

Bicubical type theory. Most closely related to our work is the paper of Weaver and Licata [WL20]. Here, Weaver and Licata consider a variant of STT based on two layers of cubical type theory and construct a directed-univalent universe in this setting. Their system, BCTT, uses two distinct interval types: one to account for homotopy type theory and a further layer for the directed interval. Bicubical type theory is therefore to TT_{\square} as cubical type theory is to HoTT. Moreover, the approach used by op. cit. to construct their universe directly inspired our own approach. In particular, the definition of amazing covariance and our directed glue type are derived from closely related constructions in BCTT. Moreover, our *directed homomorphism principle* is elaborating on an idea proposed by Weaver and Licata [WL20].

The two systems, $\mathbb{T}\mathbb{T}_{\square}$ and BCTT, differ in a number of ways. Most importantly, it is conjectured that BCTT can be formally presented¹⁶ so as to enjoy canonicity and normalization. On the other hand, $\mathbb{T}\mathbb{T}_{\square}$ certainly does not satisfy canonicity. Thus, BCTT is likely better suited for “programming” with directed univalence. However, BCTT’s categories and groupoids are not expected to be adequate for ordinary ∞ -categories or ∞ -groupoids and so it is not obvious that it can be used for developing synthetic ∞ -category theory.

More fundamentally, while they also work within an internal language and we draw on their overall strategy in Section 6, theirs is the internal *extensional* type theory of $\text{PSh}(\square_{\text{undirected}} \times \square_{\text{directed}})$ and so they must not only construct \mathcal{S} but also the model of base HoTT around it. This substantially complicates some of their constructions; their versions of e.g., covariance, Gl and so on include details that are automatically handled when working pervasively with HoTT. This model falsifies Axiom 8 and so they must introduce an additional set of axioms (the cobar modality) work around this. We believe both approaches to directed type theory warrant further consideration to (1) study our results on top of base cubical type theory rather than HoTT and (2) to translate our new results to their setting. In particular, op. cit. proves only that \mathcal{S} is directed univalent and does not prove e.g. Lemma 6.16 but we believe our proof, along with those results in Section 7, can be translated.

Other closely related type theories. While not about directed type theory, Myers and Riley [MR23] also consider a HoTT for simplicial spaces. We drew inspiration for some of our axioms (e.g. Axiom 8) from them and expect their other principles will prove useful to STT. Furthermore, Cherubini, Coquand, and Hutzler [CCH23] formulated a version of Axiom 10 to study synthetic algebraic geometry which led us to its inclusion in $\mathbb{T}\mathbb{T}_{\square}$. Finally, Riley [Ril24] presents a type theory with a single amazing right adjoint whose syntax is well-adapted for this situation. We hope that op. cit. can be generalized for $\mathbb{T}\mathbb{T}_{\square}$ to yield more usable syntax.

8.2 Future work

We isolate three key directions for future work. First, we wish to extend the experimental proof assistant Rzk [Kud23] with the minimum level of modal reasoning (e.g., at least $\langle b \mid - \rangle$, $\langle \# \mid - \rangle$ and $\langle \text{op} \mid - \rangle$) to properly axiomatize and work with \mathcal{S} as constructed in this paper. We hope to then use this to mechanize Section 7. Related to this, we hope to give a constructive model of $\mathbb{T}\mathbb{T}_{\square}$ to give a computational justification of our axioms. We expect this to contribute to a version of $\mathbb{T}\mathbb{T}_{\square}$ with canonicity and normalization [Aag+22; Gra22].

Second, in forthcoming work [GWB25b] we have generalized our construction of \mathcal{S} to construct the category of (small) categories Cat and proven that it is suitably directed univalent [CN22]. While modalities were required to construct \mathcal{S} , they are required to *state* the properties of Cat ; directed univalence becomes $\langle b \mid \parallel \rightarrow \text{Cat} \rangle \simeq \langle b \mid \sum_{A:\text{Cat}} \sum_{B:\text{Cat}} A \rightarrow B \rangle$ because homomorphisms from A to B must be the *groupoid* of the category of functors $A \rightarrow B$, not the category. Aside from this, our results scale to this more general setting.

Finally, while we discussed presheaf categories in Section 7, we avoided describing the Yoneda embedding $C \times \langle \text{op} \mid C \rangle \rightarrow \mathcal{S}$. While it is possible to construct this operation, it requires one additional modality (the twisted arrow construction) and, for reasons of space, we have regrettably chosen to omit it in the present work. In subsequent work [GWB25c]¹⁷ we have detailed this additional modality along with the resulting definition of the Yoneda embedding. Using this in

¹⁶Weaver and Licata [WL20] do not give a definition of BCTT but instead describe the intended model for any such situation. Their model is, however, constructive and so it is conjectured that such a definition would satisfy canonicity.

¹⁷The chronology of this work is somewhat confused. The construction of \mathcal{S} was completed prior to the cited work on the Yoneda embedding. However, publication timing has meant that this second paper was published first.

conjunction with our work on full subcategories, we are able to prove various important results e.g. that \mathcal{S} is cocomplete.

In additional forthcoming work [GWB25a] we are discussing cocompleteness and show the category of spectra to be stable.

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