AN ADDENDUM TO SECTION 7 OF *FIRST STEPS IN SYNTHETIC GUARDED DOMAIN THEORY*

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The following is a brief explanation of how to fill a small gap in the proof of Theorem 7.5 in Birkedal et al. [2012] that I couldn't originally see how to fill. The crux of the issue is that when symmetrizing a functor which takes two arguments: $F : \mathbb{D} \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to \mathbb{C}$, it is only reasonable to set $\widetilde{F}$ to the following type.

$$\widetilde{F} : \mathbb{D}^{\mathsf{op}} \times \mathbb{D} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}^{\mathsf{op}} \times \mathbb{C}$$

For which the definition of the object half of the new functor is:

$$\widetilde{F}(D_1, D_2, C_1, C_2) = (F(D_1, C_2, C_1), F(D_2, C_1, C_2))$$

This alteration means that the lemmas provided in Birkedal et al. [2012] must be generalized slightly to handle this.

Lemma 1 (Fixed Points are Initial)**.** *If F is a locally contractive functor $F : \mathbb{C} \to \mathbb{C}$ and I is an object such that $\iota : F(I) \cong I$ then $\iota : F(I) \to I$ forms an initial algebra for F.*

Proof. In order to show that this algebra is initial, suppose we have some $g : F(A) \to A$. We define the unique algebra homomorphism as the unique fixed point to the following contractive function.

$$h \mapsto g \circ F(h) \circ \iota^{-1}$$

This map is contractive by the assumption that the morphism component of F is witnessed by contractive maps. This is obviously an algebra homomorphism. Any algebra homomorphism, h', must satisfy the equation

$$h' \circ \iota = g \circ F(h') \iff h' = g \circ F(h') \circ \iota^{-1}$$

Therefore any algebra homomorphism is a fixed point of the above equation and by the uniqueness of fixed points $h' = h$. □

Lemma 2. *The choice of initial maps from Lemma 1 may be given as a family of maps. Explicitly, if $\iota : F(I) \cong I$ then, for any Z, there is a $f_z : Z^{F(Z)} \to Z^I$ satisfying the following formula internally.*

$$\forall g : Z^{F(Z)}.\ f_z(g) \circ \iota = g \circ F(f_z(g))$$

Proof. The construction of these families of maps is trivial. The map for some Z is given by

$$f_z = \lambda g : Z^{F(Z)}. \mathsf{fix}(\lambda h : Z^I. g \circ F(h) \circ \iota^{-1})$$

The formula follows in the internal language by the fact that in the internal language we have that a map $\mathsf{fix}(-)$ such that $\mathsf{fix}(f) = f(\mathsf{fix}(f))$ when f is contractive. This map may be constructed using unique choice, fixed points are unique so we can

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construct a functional from the internal property that they exist (a proof that unique choice holds in any topos may be found in Theorem II.5.9 of Lambek and Scott [1988]). The proof that we have unique fixed points is given by Theorem 2.9 [Birkedal et al., 2012]. $\square$

We now turn to the part which actually differs. A number of the small lemmas given in the paper must be given explicitly with regards to a parameterization in order to be sufficiently general to compose in the final proof.

Lemma 3 (Parameterized Initial Algebras). *If $F : \mathbb{D} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$, then for any $A, B \in \mathbb{D}$ if there is an $X, Y \in \mathbb{C}$ such that*

$$\iota_2 : F(A, X, Y) \cong Y \qquad \iota_1 : F(B, Y, X) \cong X$$

Then (X, Y) is an initial algebra of $\widetilde{F}(A, B, -, -) : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}^{\mathsf{op}} \times \mathbb{C}$. Furthermore, if $A \cong B$ then $X \cong Y$.

Proof. Under these assumptions we may construct an isomorphism $(\iota_1, \iota_2) : \widetilde{F}(A, B, X, Y) \to (X, Y)$. By Lemma 1 then this tells us that (ι_1, ι_2) is an initial algebra. Moreover, if $A \cong B$ then we may easily construct an initial algebra structure on (Y, X) such that in this case $X \cong Y$. $\square$

Lemma 4. *For all $F : \mathbb{D} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$, if for some $A, B \in \mathbb{D}$ there is an $X, Y \in \mathbb{C}$ such that (X, Y) is an initial algebra for $\widetilde{F}(A, B, -, -)$, then (Y, X) is an initial algebra for $\widetilde{F}(B, A, -, -)$ in $\mathbb{C} \times \mathbb{C}^{\mathsf{op}}$.*

Proof. First, if (X, Y) is an initial algebra for $\widetilde{F}(A, B, -, -)$ then we have $i_1 : X \to F(A, Y, X)$ and $i_2 : F(B, X, Y) \to Y$ satisfying the appropriate universal property.

An algebra for $\widetilde{F}(B, A, -, -)$ in $\mathbb{C}^{\mathsf{op}} \times \mathbb{C}$ is some (W, Z) equipped with a map $g : \widetilde{F}(B, A, W, Z) \to (W, Z)$. That is, a map $g_1 : F(A, Z, W) \to W$ and $g_2 : Z \to F(B, W, Z)$. This is precisely the data required for an algebra for (Z, W) on $\widetilde{F}(A, B, -, -)$ in $\mathbb{C}^{\mathsf{op}} \times \mathbb{C}$ so there's a unique map $(h_1, h_2) : (X, Y) \to (Z, W)$. It is easy to observe that $(h_2, h_1) : (Y, X) \to (W, Z)$ in $\mathbb{C} \times \mathbb{C}^{\mathsf{op}}$ and that this constitutes an algebra homomorphism for $F(B, A, -, -)$. Uniqueness is given by the fact that all algebra homomorphisms arise in this way. $\square$

This lemma goes through unchanged.

Lemma 5. *Suppose that $F : \mathbb{D} \times \mathbb{C} \to \mathbb{C}$ and for any $D \in \mathbb{D}$ that $F(D, -)$ has an initial algebra, $f_D : F(D, I_D) \to I_D$, then there exists a functor $\mu F : \mathbb{D} \to \mathbb{C}$ such that for all $D \in \mathbb{D}$, $\mu F(D)$ is the initial algebra for $F(D, -)$ (given by an appropriate family of morphisms internally witnessing initiality as in Lemma 2).*

Moreover, if F is contractive in $\mathbb{D}$ then so is μF.

Proof. Define the object part of $(\mu F)(D) = I_D$. Then, for a morphism $g : D \to D'$ define $(\mu F)(g)$ to be the unique algebra homomorphism fitting into the diagram below.

$$\begin{tikzcd}
F(D, I_D) \arrow[r, "F(D,h)"] \arrow[dd, "f_D"'] & F(D, I_{D'}) \arrow[d, "F(g, I_{D'})"] \\
 & F(D', I_{D'}) \arrow[d, "f_{D'}"] \\
I_D \arrow[r, "h"'] & I_{D'}
\end{tikzcd}$$

Then this gives the desired morphism $(\mu F)(g) = h$. The functoriality of this choice is a direct result of the fact that h is unique with the property of being a homomorphism between these two algebras.

The morphism part of this functor can be expressed as an indexed family of maps by composing the given family of maps for choosing initial algebras with the map $Y^X \to I_Y^{F(X,I_Y)}$ using the morphisms associated with F as well as f_Y. This choice is contractive if F is contractive since we compose with the strength of F. □

Lemma 6 (Unparameterized Mixed Variance Fixed Points)**.** *For any $F : \mathbb{D} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ and $A, B \in \mathbb{D}$, if $\mathbb{C}$ has solutions to contractive functors then there exists an X and a Y such that*

$$F(A, X, Y) \cong Y \qquad F(B, Y, X) \cong X$$

Proof. Define $G = F(A, -, -) : \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ and also define $H = \mu F(B, -, -) : \mathbb{C}^{\mathsf{op}} \to \mathbb{C}$ using the construction given in Lemma 5 sending each $C \in \mathbb{C}^{\mathsf{op}}$ to the fixed point of $F(B, C, -)$.

Set X as the fixed point of $C \mapsto G(H(C), C)$ (here viewing $H : \mathbb{C} \to \mathbb{C}^{\mathsf{op}}$) and set $Y = H(X)$. Then we have that $X \cong G(H(X), X) = F(A, H(X), X) = F(A, Y, X)$ and $Y = H(X) \cong F(B, X, Y)$. □

Theorem 7 (Functorially Parameterized Fixed Points)**.** *Suppose that $F : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C})^n \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to \mathbb{C}$ and is contractive in the final pair of $\mathbb{C}^{\mathsf{op}} \times \mathbb{C}$. Then there exists a functor $\mathsf{fix}(F)$ such that*

$$\mathsf{fix}(F) : (\mathbb{C}^{\mathsf{op}} \times \mathbb{C})^n \to \mathbb{C}$$

and such that

$$F \circ \left\langle 1, \widetilde{\mathsf{fix}(F)} \right\rangle \cong \mathsf{fix}(F)$$

Moreover, if F is contractive in all pairs then so is $\mathsf{fix}(F)$.

Proof. Set $\mathbb{D} = (\mathbb{C}^{\mathsf{op}} \times \mathbb{C})^n$, then $F : \mathbb{D} \times \mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}$ and therefore Lemma 6 applies to give us the preconditions to Lemma 3 so we have a choice of initial algebras for $\widetilde{F}(A, B, -, -)$, $(X_{A,B}, Y_{A,B})$ for any $A, B \in \mathbb{D}$ such that if $A \cong B$ then $X_{A,B} \cong Y_{A,B}$. Recall that $\widetilde{F}$ has the type:

$$\widetilde{F} : (\mathbb{D}^{\mathsf{op}} \times \mathbb{D}) \times (\mathbb{C}^{\mathsf{op}} \times \mathbb{C}) \to (\mathbb{C}^{\mathsf{op}} \times \mathbb{C})$$

This is in the shape for Lemma 5 where we view the $\mathbb{D}$ of this lemma as our $\mathbb{D}^{\mathsf{op}} \times \mathbb{D}$ and $\mathbb{C}$ as our $\mathbb{C}^{\mathsf{op}} \times \mathbb{C}$. This gives us a functor

$$\mu\widetilde{F} : \mathbb{D}^{\mathsf{op}} \times \mathbb{D} \to \mathbb{C}^{\mathsf{op}} \times \mathbb{C}$$

This functor is equipped with the property that $\mu\widetilde{F}(A, B) = (X_{A,B}, Y_{A,B})$. Observe that since $\mathbb{D} = (\mathbb{C}^{\mathsf{op}} \times \mathbb{C})^n$ there is a functor aspirationally named $\Delta : \mathbb{D} \to \mathbb{D}^{\mathsf{op}} \times \mathbb{D}$ defined by

$$\Delta(\overrightarrow{C^o}, \overrightarrow{C}) = ((\overrightarrow{C}, \overrightarrow{C^o}), (\overrightarrow{C^o}, \overrightarrow{C}))$$

Finally, compose with the Δ and the projection:

$$\mathsf{fix}(F) = \pi_2 \circ \mu\widetilde{F} \circ \Delta : \mathbb{D} \to \mathbb{C}$$

To get the final property desired, suppose we have some $D = (\overrightarrow{C^o}, \overrightarrow{C}) \in (\mathbb{C}^{\mathsf{op}} \times \mathbb{C})^n$. As a small piece of notation, write D^{op} for $(\overrightarrow{C}, \overrightarrow{C^o})$. Then notice that (if we denote the initial algebra constructed above for $\widetilde{F}(D^{\mathsf{op}}, D, -, -)$ as (X, Y)) the following holds.

$$\begin{aligned}
\mathsf{fix}(F)((\overrightarrow{C^o}, \overrightarrow{C})) &= (\pi_2 \circ \mu\widetilde{F} \circ \Delta)(D) \\
&= (\pi_2 \circ \mu\widetilde{F})(D^{\mathsf{op}}, D) \\
&= \pi_2(X, Y) && \mu\widetilde{F} \text{ is the initial algebra at every point} \\
&= Y
\end{aligned}$$

Now on the other hand, if we consider the left-hand side of the equation we have

$$\begin{aligned}
(F \circ \left\langle 1, \widetilde{\mathsf{fix}(F)} \right\rangle)(\overrightarrow{C^o}, \overrightarrow{C}) &= F(D, \mathsf{fix}(F)(D^{\mathsf{op}}), \mathsf{fix}(F)(D)) \\
&= F(D, \mathsf{fix}(F)(D^{\mathsf{op}}), Y) \\
&\cong F((\overrightarrow{C^o}, \overrightarrow{C}), X, Y) && \text{This step is justified below.} \\
&\cong Y
\end{aligned}$$

With the last inference following by the results of Lemma 6 which was used above to construct X and Y. We have also made use of here that if (X, Y) is an initial algebra for $\widetilde{F}(D^{\mathsf{op}}, D, -, -)$ viewed as $\mathbb{C}^{\mathsf{op}} \times \mathbb{C} \to \mathbb{C}^{\mathsf{op}} \times \mathbb{C}$ then (Y, X) is an initial algebra for $\widetilde{F}(D, D^{\mathsf{op}}, -, -)$ viewed as $\mathbb{C} \times \mathbb{C}^{\mathsf{op}} \to \mathbb{C} \times \mathbb{C}^{\mathsf{op}}$. This follows precisely from Lemma 4. Explicitly worked out:

$$\begin{aligned}
\mathsf{fix}(F)(D^{\mathsf{op}}) &= (\pi_2 \circ \mu\widetilde{F} \circ \Delta)(D^{\mathsf{op}}) \\
&= (\pi_2 \circ \mu\widetilde{F})(D, D^{\mathsf{op}}) \\
&\cong \pi_2(Y, X) && \mu\widetilde{F}(D, D^{\mathsf{op}}) \text{ is the initial algebra in } \mathbb{C} \times \mathbb{C}^{\mathsf{op}} \\
&= X
\end{aligned}$$

Note further that the second to last line is merely an isomorphism: this is because we are only guaranteed to choose some initial algebra, not (Y, X) in particular. The uniqueness of initial algebras up to isomorphism is sufficient to determine the fact that we wanted however. One can view this as baking in the final line of reasoning for Lemma 3 and it explains why we never made explicit use of this fact.

This demonstrates the desired isomorphism.

The fact that $\mathsf{fix}(F)$ is contractive in all pairs is a result of the fact that $\mu\widetilde{F}$ is contractive in its first argument if F is. $\square$

References

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