ADJOINT MODALITIES IN MTT

DANIEL GRATZER

Abstract. We record several results about the behavior of adjoint modalities in MTT. In particular, we show that internal adjunctions can be used to recover stronger rules, similar to Birkedal et al. [Bir+20].

We explore the relationship between dependent right adjoints and a weak dependent right adjoint whose left adjoint also internalizes as a modality. We argue that these *internal right adjoints* exhibit many of the nice properties of dependent right adjoints. Together with recent results of Gratzer et al. [Gra+22], we argue that restricting to weak dependent right adjoints poses little issue in practice.

Remark 1. Andreas Nuyts has since pointed out to us that Theorems 1 and 2 are already present in the work by Nuyts and Devriese work on *transpension types* in MTT [ND21]. In particular, Propositions 3.3 and 3.4 establish these results for extensional MTT. We keep have preserved this note for expository purposes and to discuss the validity of these results in various subsystems of extensional MTT.

1. Internal adjoints

Let us consider the mode theory $\mathcal{M}$ which contains two modalities $\mu : n \longrightarrow m$ and $\nu : m \longrightarrow n$ together with 2-cells witnessing $\nu \dashv \mu$. Explicitly, there are 2-cells $\eta : \mathsf{id}_m \longrightarrow \mu \circ \nu$ and $\epsilon : \nu \circ \mu \longrightarrow \mathsf{id}_n$ satisfying the triangle equations:

(1)
$$\begin{array}{ccc} \mu & \xrightarrow{\eta \star \mathsf{id}_\mu} & \mu \circ \nu \circ \mu \\ & \searrow^{\mathsf{id}_\mu} & \downarrow{\scriptstyle \mathsf{id}_\mu \star \epsilon} \\ & & \mu \end{array}$$

(2)
$$\begin{array}{ccc} \nu & \xrightarrow{\mathsf{id}_\nu \star \eta} & \nu \circ \mu \circ \nu \\ & \searrow^{\mathsf{id}_\nu} & \downarrow{\scriptstyle \epsilon \star \mathsf{id}_\nu} \\ & & \nu \end{array}$$

Mode theories of this shape were considered to some extent in Gratzer et al. [Gra+20] and they have shown that ν behaves like a left adjoint internal to MTT and that e.g. it preserves certain colimits.

Date: Saturday 18th February, 2023.
This note was originally titled *Self-adjoint modalities in MTT*.

We consider the behavior of the right adjoint μ. We first observe that the action of μ on context can be encoded to through ν:

Theorem 1. *For any context* $\Gamma\ \mathsf{cx}$*, modality* $\xi : o \longrightarrow m$*, and* $\Gamma.\{\xi\} \vdash A$*:*

$$\Gamma.(\xi \mid A).\{\mu\} \cong \Gamma.\{\mu\}.(\nu \circ \xi \mid A^{\eta \star \xi})$$

Proof. First, we observe that because $-.\{-\}$ is a 2-functor, it preserves adjoints. Therefore, the $-.\{\nu\} \dashv -.\{\mu\}$ as functors on categories of contexts.

We will first argue that $\Gamma.(\xi \mid A).\{\mu\}$ and $\Gamma.\{\mu\}.(\nu \circ \xi \mid A^{\eta \star \xi})$ are isomorphic as they represent the same functor. To this end, we make use of universal property of context extension in MTT: a substitution $\Delta_0 \longrightarrow \Delta_1.(\xi \mid A)$ is determined by (1) a substitution $\delta : \Delta_0 \longrightarrow \Delta_1$ and (2) a term $\Delta_0 \vdash M : A[\delta.\{\xi\}]$ [Gra+20].

Fix a context Δ in mode n. Using the above universal property along with transposition, a substitution $\Delta \longrightarrow \Gamma.\{\mu\}.(\nu \circ \xi \mid A^{\eta \star \xi})$ is determined by (1) a substitution $\gamma : \Delta.\{\nu\} \longrightarrow \Gamma$ and (2) a term $\Delta.\{\nu \circ \xi\} \vdash M : A^{\eta \star \xi}[\widehat{\gamma}.\{\nu \circ \xi\}]$ naturally Δ. Unfolding the definition of transposition, $A^{\eta \star \xi}[\widehat{\gamma}.\{\nu \circ \xi\}]$ is simply $A[\gamma.\{\xi\}]$.

Next, a substitution $\Delta \longrightarrow \Gamma.(\xi \mid A).\{\mu\}$ is determined by (1) a substitution $\gamma : \Delta.\{\nu\} \longrightarrow \Gamma$ and (2) a term $\Delta.\{\nu \circ \xi\} \vdash M : A[\gamma.\{\xi\}]$ naturally in Δ.

The two contexts are therefore isomorphic by the Yoneda lemma. □

Theorem 2. *Given any context* $\Gamma\ \mathsf{cx}$ *and* $\Gamma.\{\mu\} \vdash A$ *when* $\nu \dashv \mu$*, there is a pair of substitutions*

$$\gamma^{\rightarrow} : \Gamma.(\mu \mid A) \longrightarrow \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle)$$
$$\gamma^{\leftarrow} : \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \longrightarrow \Gamma.(\mu \mid A)$$

Moreover, $\gamma^{\leftarrow} \circ \gamma^{\rightarrow} = \mathsf{id}$ *and, if one assumes extensional equality,* $\gamma^{\rightarrow} \circ \gamma^{\leftarrow} = \mathsf{id}$.

Proof. One direction of this isomorphism holds regardless of the precise properties of μ:

$$\gamma^{\rightarrow} \triangleq {\uparrow}.\mathsf{mod}_\mu(\mathbf{v}_0) : \Gamma.(\mu \mid A) \longrightarrow \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \tag{3}$$

The inverse direction is more subtle:

$$\gamma^{\leftarrow} \triangleq {\uparrow}.M : \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \longrightarrow \Gamma.(\mu \mid A) \tag{4}$$

Here, M must be a term of the following type:

$$\Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle).\{\mu\} \vdash M : A[{\uparrow}.\{\mu\}]$$

In order to define this, consider the following term:

$$\frac{\begin{array}{c}\Gamma.(\mathsf{id}_n \mid \langle \mu \mid A \rangle).\{\mu \circ \nu\} \vdash \mathbf{v}_0^\eta : \langle \mu \mid A[\{\eta \star \mathsf{id}_\mu\}] \rangle \\ \Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle).\{\mu\}.(\nu \circ \mu \mid A) \vdash \mathbf{v}_0^\epsilon : A[{\uparrow}.\{\nu\}]\end{array}}{\Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle).\{\mu\} \vdash M \triangleq \mathsf{let}_\nu\ \mathsf{mod}_\mu(\_) \leftarrow \mathbf{v}_0^\eta\ \mathsf{in}\ \mathbf{v}_0^\epsilon : A[{\uparrow}.\{\mu\}]}$$

By computation, we immediately have $\gamma^{\leftarrow} \circ \gamma^{\rightarrow} = \mathsf{id}$. In the reverse direction, we must show that the following terms are definitionally equivalent

$$\Gamma.(\mathsf{id}_m \mid \langle \mu \mid A \rangle) \vdash \mathbf{v}_0 = \mathsf{mod}_\mu(\mathsf{let}_\nu\ \mathsf{mod}_\mu(\_) \leftarrow \mathbf{v}_0^\eta\ \mathsf{in}\ \mathbf{v}_0^\epsilon) : \langle \mu \mid A[{\uparrow}.\{\mu\}] \rangle \tag{5}$$

This equation is true *propositionally*, by performing induction on $\mathbf{v}_0$. Therefore, in the presence of extensional equality this holds definitionally as well. □

With this result to hand, we define $\mathsf{unmod}_\mu(M)$ as follows:

$$\frac{\Gamma \vdash M : \langle \mu \mid A \rangle}{\Gamma.\{\mu\} \vdash \mathsf{unmod}_\mu(M) = \mathbf{v}[\gamma^{\leftarrow}.\{\mu\} \circ \mathsf{id}.M.\{\mu\}] : A}$$

Remark 2. We could alternatively formulate $\mathsf{unmod}_\mu(-)$ with the following rule:

$$\frac{\Gamma.\{\nu\} \vdash M : \langle \mu \mid A \rangle}{\Gamma \vdash \mathsf{unmod}_\mu(M) : A[\{\epsilon\}]}$$

The two formulations are inter-derivable. The one we gave above is more commonly found in the literature [Bir+20], but this alternative can be taken as primitive without disrupting substitution.

Lemma 3. *If* $\Gamma.\{\mu\} \vdash M : A$ *then* $\mathsf{unmod}_\mu(\mathsf{mod}_\mu(M)) = M$

Proof. We must show the following:

$$\mathbf{v}[(\gamma^{\leftarrow} \circ \mathsf{id}.\mathsf{mod}_\mu(M)).\{\mu\}] = M$$

To this end, let us first rewrite $\mathsf{id}.\mathsf{mod}_\mu(M)$ as $\uparrow.\mathbf{v} \circ \mathsf{id}.M$. We then observe that this is precisely $\gamma^{\rightarrow} \circ \mathsf{id}.M$ whence we have the following:

$$\begin{aligned}&\mathbf{v}[(\gamma^{\leftarrow} \circ \mathsf{id}.\mathsf{mod}_\mu(M)).\{\mu\}]\\ &= \mathbf{v}[(\gamma^{\leftarrow} \circ \gamma^{\rightarrow} \circ \mathsf{id}.M).\{\mu\}]\\ &= M\end{aligned}$$ □

Lemma 4. *There is a propositional equality:*

$$(x : \langle \mu \mid A \rangle) \to \mathsf{Id}_{\langle \mu \mid A \rangle}(\mathsf{mod}_\mu(\mathsf{unmod}_\mu(x)), x)$$

Proof. Modal induction on x reduces this to Lemma 3. □

Remark 3. Note that Theorem 2 and Lemmas 3 and 4 only requires a fraction of the full elimination rule MTT provides. In particular, it is only necessary to use id or ν as a framing modality.

Remark 4. Note that, in particular, if we consider a mode theory with a single self-adjoint modality $\mu = \nu$, these results ensure that (extensional) MTT coincides with the type theory proposed by Riley, Finster, and Licata [RFL21]. In standard MTT, there is a slight distinction with *op. cit.* providing a definitional η law for modalities where Lemma 4 is merely propositional.

References

[Bir+20] Lars Birkedal, Ranald Clouston, Bassel Mannaa, Rasmus Ejlers Møgelberg, Andrew M. Pitts, and Bas Spitters. "Modal dependent type theory and dependent right adjoints". In: *Mathematical Structures in Computer Science* 30.2 (2020), pp. 118–138. DOI: 10.1017/S0960129519000197 (cit. on pp. 1, 3).

[Gra+20] Daniel Gratzer, G.A. Kavvos, Andreas Nuyts, and Lars Birkedal. *Multimodal Dependent Type Theory*. Extended version of the LICS paper by the same name. 2020. URL: https://arxiv.org/abs/2011.15021 (cit. on pp. 1, 2).

[Gra+22] Daniel Gratzer, Evan Cavallo, G. A. Kavvos, Adrien Guatto, and Lars Birkedal. "Modalities and Parametric Adjoints". In: *ACM Trans. Comput. Logic* 23.3 (Apr. 2022). ISSN: 1529-3785. DOI: 10.1145/3514241 (cit. on p. 1).

[ND21] Andreas Nuyts and Dominique Devriese. *Transpension: The Right Adjoint to the Pi-type*. 2021. URL: https://arxiv.org/abs/2008.08533 (cit. on p. 1).

[RFL21] Mitchell Riley, Eric Finster, and Daniel R. Licata. *Synthetic Spectra via a Monadic and Comonadic Modality*. 2021 (cit. on p. 3).