A modal deconstruction of Löb induction

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What are we trying to do?

Claim: recursive equations are the bedrock of denotational semantics for PLs.

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We can even prove it formally verify this claim in type theory!

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recursive equations & type theory do not mix.

Guarded type theory: the basic components

A proposed answer: guarded type theory!

- fix is safe as long as we "do work" before recurring
- Use a unary type constructor (modality) to crystallize this discipline.

We arrive at the basic components of guarded recursion:

$$\blacktriangleright : \mathcal{U} \to \mathcal{U} \qquad\qquad \mathsf{loeb} : (A : \mathcal{U}) \to (\blacktriangleright A \to A) \to A$$

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Lots of ways to structure this [Nak00; AM13; BM13; Møg14; BGM17; GB22; KMV22]

Now the semantics of ▶

Guarded recursion $\sim$ proof-relevant step-indexing (via $\mathbf{PSh}(\omega)$)

$$\begin{array}{ccc} & & \cdots \\ & & \downarrow \\ & & X_2 \\ & & \downarrow \\ [\![X]\!] & = & X_1 \\ & & \downarrow \\ & & X_0 \end{array} \qquad\qquad \begin{array}{ccc} & & \cdots \\ & & \downarrow \\ & & X_1 \\ & & \downarrow \\ [\![\blacktriangleright X]\!] & = & X_0 \\ & & \downarrow \\ & & \{\star\} \end{array}$$

Types $\sim$ Time-indexed sets

Terms $\sim$ Time-preserving maps

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Are we done?

A few things are left to nail down:

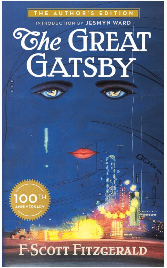
1. $\blacktriangleright$ is part of a whole family of modal operators, what about them?
2. $\blacktriangleright$/loeb are sound, but what about canonicity, normalization, etc.

The challenge: how do we have a full dependent type theory with (1) and (2).

Contributions: Gatsby

Turns out, we can reduce this to a question of modalities:

- We show that adding a new principle *about modalities* allows us to derive Löb.
- Crucially rely on univalence/HoTT in a few places...
- All told, obtain a new framework for guarded recursion: Gatsby.



Definitely satisfies (1) and have good evidence that (2) is also true.

Guarded type theory: modalities

One modality $\blacktriangleright$ (also written $\langle \ell \mid - \rangle$ for uniformity) is good, surely more is better!

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\begin{tikzcd}
t \arrow[loop left, "{\ell, e}"] \arrow[rr, bend left, "\epsilon_0"] \arrow[rr, bend right, "\gamma" description] & & s \arrow[ll, "\delta" description] \arrow[loop right, "\top"]
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- Two classes of types: t for those which are guarded, s for standard types
- More modalities $\sim$ more control over "how fast" terms produce answers.
- Ex. $\langle e \mid A \rangle$ is "an A which has already done work" so $\langle e \mid \langle \ell \mid A \rangle \rangle \simeq A$.

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Relatively new: adding modalities & univalence is cheap [Gra+20; Aag+22].

Guarded type theory: Löb induction

Theorem

No matter the combination of modalities, we cannot define $\mathsf{loeb} : (\langle \ell \mid A \rangle \to A) \to A$.

Proof.

Nothing allows us to internally prove $\langle \ell \mid A \rangle \neq A$. □

Pictured: Gatsby longing for Löb induction.

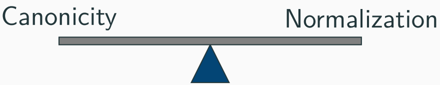
Löb induction in type theory

Theorem (Gratzer and Birkedal [GB22])

If loeb *computes then type-checking is undecidable.*

Lemma

If loeb *never computes then canonicity fails.*



- Clocked type theory [BGM19; KMV22], let loeb compute only sometimes...
- Adding loeb is hard, I would prefer not to.

Big idea: move the goal posts

Let's play a new game:

$$💣 = \left\| \sum_{n:\mathbf{Nat}} \blacktriangleright^{n} \bot \right\|$$

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$\|-\|$ is propositional truncation, so read $\exists n : \mathbf{Nat}.\, \blacktriangleright^{n}\bot$

Holds in $\mathbf{PSh}(\omega)$; choice of n doesn't have to agree at each step!

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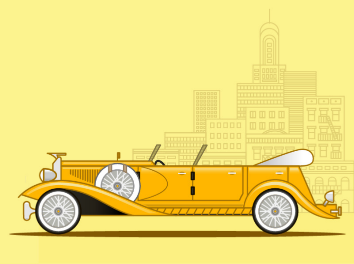
Key Idea

💣 is a *doomsday proposition*; hypothesizes everything will collapse... eventually.

Pictured: Well-known symbolism for 💣.



From existential dread to Löb induction

Theorem

If 💣 *then every operation* $f : \langle \ell \mid A \rangle \to A$ *has a* ***unique*** *guarded fixed-point:*

$$💣 \to \mathsf{isContr}\left(\sum_{a:A} \text{“}a \text{ is a guarded fixed-point of } f\text{”}\right)$$

Proof Sketch.

The goal is an (h-)proposition, so ignore $\|-\|$ and use induction on $n : \mathbf{Nat}$. $\square$

Univalence Alert

Relies on HoTT's more *semantic* notion of proposition/truncation.

Goal posts shifted... Now what to do with 💣?

If 💣 were true, we'd be done... but this breaks decidable type-checking [GB22]

Key Idea

It suffices to find a supply of (very cynical) types which *believe* 💣 to be true.

Definition

A is accessible if the canonical map $A \to ($💣$\to A)$ is an equivalence.

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Theorem

If A is accessible then $\mathrm{loeb} : (\blacktriangleright A \to A) \to A$

Not every type is accessible (for instance, not 💣!) but a lot of them are...

Accessible types

Theorem (Rijke, Shulman, and Spitters [RSS20])

Accessible types form a reflective subuniverse closed under Π, Σ, $=$, $\mathbf{1}$, *and* $\mathcal{U}_{\text{💣}}$.

Univalence Alert

Crucial use of univalence to show that $\mathcal{U}_{\text{💣}} = \sum_{A:\mathcal{U}} \mathrm{isAcc}(A)$ is accessible.

This is all well and good, but we want some non-trivial base types!

Enter $\top$

To move further, we turn back to our new modality: $\top : s \longrightarrow s$:

Key Idea

Add one new rule to force $\langle \top \mid A \rangle \simeq \mathbf{Unit}$.

Immediate consequence: $\langle \epsilon_0 \mid \langle \ell \mid A \rangle \rangle \simeq \langle \top \mid A \rangle$

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$$\frac{\text{For the } \Gamma \text{ and } \vdash \text{ rules forcing } \langle \top \mid A \rangle \simeq \textbf{Unit}}{\text{See the paper!}}$$

The world's hardest working unit type

Remarkably, $\top$'s existence yields more accessible types!

Theorem

- *If* $\langle \ell \mid \mathrm{isAcc}(A) \rangle$ *then* $\langle \ell \mid A \rangle$ *is accessible.*
- *For any s type (i.e., not guarded)* A*,* $\langle \delta \mid A \rangle$ *is accessible.*

Corollary

Being accessible is closed under $\mathbf{0}$, $+$, $\mathbf{Bool}$, *and* $\mathbf{Nat}$.

Checking the scoreboard... that's pretty much all the type-formers!

The payoff

Since so many types are accessible, we can begin work exclusively with accessible types.

Theorem

Standard guarded type theory ($\square, \blacktriangleright$ etc.) has a model in accessible types.

- Since e.g., **Nat** is accessible, we can use loeb to compute concrete numbers!
- It also means that users of Gatsby never actually have to talk about 💣

A few case studies

Extra modalities & standard v guarded type separation give new opportunities:

- We define a logical relation for general references using loeb for type safety
- We construct non-guarded coinductive types using guarded recursion internally.
- Both of these prove results about non-guarded objects using guarded recursion!

Via a semantic model, an adequate proof strategy: cubical version of $\mathbf{PSh}(\omega)$.

Is Gatsby great?

We show that adding a new principle *about modalities* allows us to derive Löb.

- Crucially rely on univalence/HoTT in a few places...
- All told, obtain a new framework for guarded recursion.

Definitely satisfies (1) and have good evidence that (2) is also true.

A few words on the proof

Theorem

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The proof amounts to two crucial facts:

- $\langle e \mid \text{💣} \rangle \simeq \text{💣}$
- $\langle \epsilon_0 \mid \text{💣} \rangle \simeq \mathbf{1}$.

Proven formal modal shuffling & the fact that $\langle \top \mid A \rangle = \mathbf{1}$.

Key Idea

Novel rule means $\langle \top \mid A \rangle \neq A$ and equations between modalities then ripple out.

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