Modalities and (weak) dependent right adjoints

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What’s the goal of type theories?

Type theories do a very good job working with families of objects

- All of proofs in type theory are relative to a context
- All connectives are natural in the context, making it invisible
- A family of widgets is equivalent to a widget (in a different context)

The second point is crucial to this story!

Why does it matter that connectives are natural?

Fix some term:

$$\Phi : \prod_{n:\mathbb{N}} \sum_{m:\mathbb{N}} P(n, m)$$

If we instantiate Φ with some number, result has the following type:

$$\Phi(2) : \sum_{m:\mathbb{N}} P(2, m)$$

If instead $\Phi(2) : \sum_{m:\mathrm{Bool}} P(2, m)$, this would be a problem!

Why does it matter that connectives are natural? II

This sort of oddity does not happen precisely because Σ is natural:

$$(\Sigma_A B)[\gamma] = \Sigma_{A[\gamma]} B[(\gamma \circ \mathbf{p}).\mathbf{v}]$$

- We must have this rule for every connective
- Explicit substitutions are one possible formulation
- What matters: substitutions flow past connectives and to variables

What's the goal of modal type theories

The goal of modal type theories:

1. Add a connective—a modality—which is not stable under all substitutions
2. Produce a usable system out of what remains

The last goal is squishy, but ideally...

- should be implementable
- should be able to use it for formalization

Example 1: using modalities to capture global behavior

Modalities can help us construct models of cubical type theory:

- Orton and Pitts [OP18]: the internal language helps to model various types
- We need the tininess of $\mathbb{I}$ to model the universe:

$$\iota : (-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$$

- Two issues: neither $(-)_{\mathbb{I}}$ nor ι internalize well!
- $(-)_{\mathbb{I}}$ doesn't descend to each slice

Solution: use the global sections comonad $\square$ [Lic+18]:

$$(-)_{\mathbb{I}} : \square\mathcal{U} \to \mathcal{U}$$

NB: $\square/(-)_{\mathbb{I}}$ cannot be naively added to type theory [Shu18].

Example 2: using modalities to model features of a model

Guarded recursion gives us access to a powerful form of synthetic domain theory:

- The key primitives ($\blacktriangleright$, $\mathsf{loeb} : (\blacktriangleright A \to A) \to A$) can be axiomatized [Bir+12]
- On their own, insufficient to define certain operations (e.g., adequacy, termination)

Solution: add another $\square$ modality along with the following equivalence:

$$\square \blacktriangleright A \simeq \square A$$

How do we actually include a modality?

Including a modality is causes a lot of problems, but it's useful as well..

> **Q.** How do we actually go about including a modality?
>
> **A.** Alter contexts and substitutions to *make* the modality natural.

The result is an odd type theory, but with a modality and some substitution lemma.

We now have 25 years worth of different alterations for different types of modalities.

The goal of this talk

Our goal is threefold:

- Discuss a few of the more common alterations to make to type theory
- Consider the semantics of the resultant theories
- Argue that they can be seen as alterations on one specific form of alteration (wDRAs)

The takeaway: wDRAs are ubiquitous.

Non-goals of this talk

We have some non-goals as well:

- We won't consider substructural theories
- We'll assume that our modalities are (suitably) lex
- We will *not* assume that they're fibered

Primordial rules for modalities

All our example modalities are lex functors. Can we capitalize on this?

$$\frac{\Gamma \vdash A}{F(\Gamma) \vdash F(A)} \qquad\qquad F(A)[F(\gamma)] = F(A[\gamma])$$

$$\epsilon : F(\Gamma.A) \cong F(\Gamma).F(A) \qquad\qquad F(\mathbf{p}) = \mathbf{p} \circ \epsilon \qquad\qquad \tau : F(\mathbf{1}) = \mathbf{1}$$

Pros: easy to interpret!

Theorem

Given a display map category and a functor preserving display maps and pullbacks along them as well as $\mathbf{1}$*, we may interpret the above using local universes [LW15].*

Deriving the introduction rule

Notice that with these primitives we can derive a functorial introduction rule:

$$\frac{\Gamma \vdash M : A}{F(\Gamma) \vdash F(M) : F(A)}$$

- Consider the substitution $F(\Gamma) \vdash F(\mathrm{id}.M) : F(\Gamma.A)$
- Post-compose with ϵ to obtain $F(\Gamma) \longrightarrow F(\Gamma).F(A)$ over $F(\Gamma)$
- Use the variable rule

In total:

$$F(M) = \mathbf{v}[\epsilon \circ F(\mathrm{id}.M)]$$

Primordial modalities and substitution

The downside of this approach: no good substitution lemma.

- Fix a type $\Gamma \vdash A$ and substitution $\Delta \vdash \gamma : F(\Gamma)$
- Cannot "push" γ inside $F(A)$; don't have $\gamma = F(\gamma')$
- Result: no clear equation to associate with in general $F(A)[\gamma]$

Not as bad as $\mathbb{N}$ becoming Bool, but still prevents us from working context-agnostically.

Solution 1: Delayed substitutions

We can just accept this and move on. Obtain *delayed substitutions* [Bd00]

$$\frac{\Gamma \vdash \delta : F(\Delta) \qquad \Delta \vdash A}{\Gamma \vdash F_\delta(A)}$$

- Relatively simple, scales to many modalities
- Sometimes can give delayed substitutions slighter nicer syntax [Biz+16]
- Normalization fails; type-checking cannot be decided
- Still have to reason about substitutions directly so not very usable

Why does normalization fail?

The failure of normalization is intimately connected with the substitution law

$$\begin{tikzcd}
\Delta \arrow[rr, "\gamma"] \arrow[dr, "\delta"'] & & F(\Gamma) \vdash F(A) \\
& F(\Delta_0) \arrow[ur, "F(\gamma')"'] &
\end{tikzcd}$$

Too many possible choices of Δ_0; each choice yields a different reduction:

$$F(A)[\gamma] = F(A[\gamma'])[\delta_0]$$

Uniqueness is unreasonable.

A possible way out: a universal delay

$$\Delta \xrightarrow{\quad\quad\quad\gamma\quad\quad\quad} F(\Gamma) \vdash F(A)$$

- We can't get a unique factorization, but perhaps there's a best choice?
- In fact, we are interested in the initial factorization

A possible way out: a universal delay

$$\begin{tikzcd} \Delta \arrow[rr, "\gamma"] \arrow[dr, "\eta"'] & & F(\Gamma) \vdash F(A) \\ & F(L(\Delta)) \arrow[ur, "F(L(\gamma))"'] & \end{tikzcd}$$

- We can't get a unique factorization, but perhaps there's a best choice?
- In fact, we are interested in the initial factorization

Requiring this for all Δ and all $\Delta \vdash \gamma : \Gamma$ is equivalent to asking for $L \dashv F$.

A universal delay II

This is good enough! Can replace our delayed substitution with η; the universal delay:

$$\frac{L(\Gamma) \vdash A}{\Gamma \vdash F_\eta(A)}$$

$$\begin{aligned} F_\eta(A)[\gamma] &= F_{\eta \circ \gamma}(A) \\ &= F_{F(L(\gamma)) \circ \eta}(A) \\ &= F_\eta(A[L(\gamma)]) \end{aligned}$$

A universal delay III

What about introduction and elimination?

Note that adjunctions descend to slices: $\mathsf{Cx}/\Gamma \leftrightarrows \mathsf{Cx}/F(\Gamma)$:

$$\begin{aligned}
\mathsf{Tm}(L(\Gamma), A) &\cong \mathsf{Hom}_{\mathsf{Cx}/L(\Gamma)}(L(\Gamma), L(\Gamma).A) \\
&\cong \mathsf{Hom}_{\mathsf{Cx}/F(L(\Gamma))}(\Gamma, F(L(\Gamma).A)) \\
&\cong \mathsf{Hom}_{\mathsf{Cx}/F(L(\Gamma))}(\Gamma, F(L(\Gamma)).F(A)) \\
&\cong \mathsf{Hom}_{\mathsf{Cx}/\Gamma}(\Gamma, \Gamma.F_\eta(A)) \\
&\cong \mathsf{Tm}(\Gamma, F_\eta(A))
\end{aligned}$$

Rendered as rules:

$$\frac{L(\Gamma) \vdash M : A}{\Gamma \vdash F_\eta(M) : F_\eta(A)} \qquad\qquad \frac{\Gamma \vdash M : F_\eta(A)}{L(\Gamma) \vdash \mathsf{unmod}(M) : A}$$

Solution 2: dependent adjunctions

Small observation: we no longer need F to act on contexts, just types.

Definition (Birkedal et al. [Bir+20])

A dependent adjunction L, F consists of a functor from contexts to contexts L and a map of types $F : \mathsf{Ty}(L(\Gamma)) \longrightarrow \mathsf{Ty}(\Gamma)$ equipped with a natural bijection of terms:

$$\mathsf{Tm}(L(\Gamma), A) \cong \mathsf{Tm}(\Gamma, F(A))$$

DRATT: type theory extended with a dependent adjunction.

Models of DRATT

It's still relatively easy to cook up models of DRATT:

Theorem

Given a display map category and an endo-adjunction whose right adjoint preserves display maps, we may interpret DRATT.

- The second requirement often automatic (e.g. with presentable categories)
- All our earlier examples are DRAs

What about the elimination rule?

Returning to the elimination rule, we have the same problem as we started with!

$$\frac{\Gamma \vdash M : F(A)}{L(\Gamma) \vdash \mathsf{unmod}(M) : A}$$

Given a substitution $\Delta \vdash \gamma : L(\Gamma)$, we still don't have $\gamma = L(\gamma')$.

An ad-hoc solution

Workaround: there aren't many definable substitutions into $L(\Gamma)$.

- Scrutinize the definable substitutions and manually close the rule under them
- For instance, with just a dependent adjunction it suffices to close under weakening

$$\frac{\Gamma \vdash M : F(A)}{L(\Gamma).A_0.A_1.A_2 \ldots A_n \vdash \mathrm{unmod}(M) : A}$$

This "works", but you have to redo this for any tweak you make to F.

Analyzing DRATT

- This calculus is usable and implementable [GSB19; BGM17]
- Cooking up these elimination rules is hard work (some improvements [HP23])
- It's unclear that this can be done for more than one modality at time
- It's also unsatisfying to depend on a particular property of the syntactic model

Taming the elimination rule

Can we play the same game as we did with the formation rule?

- We ask that any substitution $\Delta \longrightarrow L(\Gamma)$ admits an initial factorization
- Unlike before, not every Δ necessarily has a map to $L(-)$
- We only require this factorization when we have a map $\Delta \longrightarrow L(\mathbf{1})$

Rephrasing: L should be a PRA:

Definition

A functor $G : \mathcal{C} \longrightarrow \mathcal{D}$ is a parametric right adjoint if the induced functor $\mathcal{C} \longrightarrow \mathcal{D}/G(\mathbf{1})$ is a right adjoint.

A new elimination rule

As before, we may now rephrase the elimination rule to use this generic case:

$$\frac{\Gamma \vdash \rho : L(\mathbf{1}) \qquad U(\Gamma, \rho) \vdash M : F(A)}{\Gamma \vdash \mathsf{unmod}(M, \rho) : A[\eta]}$$

NB: η is the unit of the parametric adjunction, it also depends on ρ.

Rationalizing PRAs

Definition

An exceptional dependent adjunction is a dependent adjunction whose left adjoint is a PRA.

- We should regard an exceptional dependent adjunction as a sort of product.
- The substitution $\Gamma \vdash \rho : L(\mathbf{1})$ is the "argument"

Theorem

For any closed type $\mathfrak{C}$, the dependent product $\prod_{\mathfrak{C}} -$ is an exceptional DRA

Rationalizing PRAs II

Revisiting the earlier rule with $F(A) = \prod_{\mathfrak{C}} A$

$$\frac{\Gamma \vdash \rho : \mathbf{1}.\mathfrak{C} \qquad \Gamma \vdash M : \prod_{\mathfrak{C}} A}{\Gamma \vdash \mathsf{unmod}(M, \rho) : A[\eta]}$$

ρ is equivalent to a term $\Gamma \vdash N : \mathfrak{C}$.

ticks: fancy syntax inspired out of this intuition [BGM17].

Relating PRAs to DRATT

In fact, this machinery gives us a way to restate our earlier adhoc results:

Theorem

The left adjoint on the syntactic model of DRATT is a PRA.

In fact, the adhoc rules exist *because* the necessary factorizations exist.

The lemmas we must prove correspond precisely to the theorem above.

Solution 3: FitchTT

All of this discussion leads us to our third modal type theory: FitchTT [Gra+22].

- Essentially, extend type theory with exceptional dependent adjunctions
- No issues incorporating multiple modalities and no messy syntactic arguments
- Conjecturally, normalization and type-checking are possible

The semantics of FitchTT are a straightforward extension of those of DRATT:

Theorem

A model of FitchTT is precisely a model of DRATT with a parametric adjunction.

Corollary

FitchTT is a conservative extension of DRATT.

The downsides of FitchTT

The downside of FitchTT is not difficult to spot:

- How many modalities of interest are right adjoints?
- Of that subset, how many are part of an exceptional DRA?
- A non-zero amount ($\Box$ and $\blacktriangleright$, for instance), but it's a big ask

It's not ideal that we are able to include roughly one of the three functors in play here!

An intermezzo with notation

Before proceeding further, it's helpful to fix some notation for multiple modalities.

- Fix a 2-category $\mathcal{M}$ of modes, modalities, and transformations between them
- The modes are less important; safe to assume there is just one
- μ, ν, ξ for modalities and α, β for 2-cells
- We'll always have composites of modalities and an identity
- We'll use $\langle \mu \mid - \rangle$[1] and $-.\{\mu\}$ for modal types and left adjoints.

We require that $-.\{\mu\}$ is a 2-functor $\mathcal{M}^{\mathsf{coop}} \longrightarrow \mathbf{Cat}$:

$$\Gamma.\{\mathsf{id}\} = \Gamma \qquad \Gamma.\{\mu\}.\{\nu\} = \Gamma.\{\mu \circ \nu\}$$

[1] My proposed notation of regular n-gons and simply increasing *n* to generalize $\square$ has not yet taken off.

Retreating from dependent adjunctions

Can we instead weaken dependent adjunctions to arrive at a workable syntax instead?

We'll split things up into two halves:

1. a judgmental structure representing the right adjoint
2. a type which *weakly* internalizes this structure

We term the resulting structure a *weak dependent adjunction*

Towards weak dependent adjunctions

A judgmental structure representing the right adjoint

Modify context extension as follows:

$$\frac{\Gamma.\{\mu\} \vdash A}{\vdash \Gamma.(\mu \mid A)\ \mathsf{cx}} \qquad \mathsf{Hom}(\Delta, \Gamma.(\mu \mid A)) \cong \sum_{\gamma : \mathsf{Hom}(\Delta,\Gamma)} \mathsf{Tm}(\Delta.\{\mu\}, A[\gamma.\{\mu\}])$$

In particular, we have $\Gamma.(\mu \mid A) \vdash \mathbf{p} : \Gamma$ and $\Gamma.(\mu \mid A).\{\mu\} \vdash \mathbf{v} : A[\mathbf{p}.\{\mu\}]$.

Towards weak dependent adjunctions II

Restricting to *sections*, this gives back a dependent adjunction-esque isomorphism:

$$\mathsf{Tm}(\Gamma.\{\mu\}, A) \cong \mathsf{Sec}\big(\mathbf{p} : \Gamma.(\mu \mid A) \longrightarrow \Gamma\big)$$

Since $\Gamma.\{\mathsf{id}\} = \Gamma$, we recover normal context extension:

$$\mathsf{Hom}(\Delta, \Gamma.(\mathsf{id} \mid A)) \cong \sum\nolimits_{\gamma : \mathsf{Hom}(\Delta, \Gamma)} \mathsf{Tm}(\Gamma, A[\gamma])$$

Towards weak dependent adjunctions III

A type weakly internalizing this structure

The formation and introduction rules will be familiar, but we'll have a much weaker elimination rule:

$$\frac{\Gamma.\{\mu\} \vdash A \,\mathsf{type}}{\Gamma \vdash \langle \mu \mid A \rangle \,\mathsf{type}} \qquad \frac{\Gamma.\{\mu\} \vdash M : A}{\Gamma \vdash \mathsf{mod}(M) : \langle \mu \mid A \rangle}$$

$$\frac{\Gamma.(\mathsf{id} \mid \langle \mu \mid A \rangle) \vdash B \,\mathsf{type} \qquad \Gamma \vdash M_0 : \langle \mu \mid A \rangle \qquad \Gamma.(\mu \mid A) \vdash M_1 : B[\mathbf{p}.\mathsf{mod}(\mathbf{v})]}{\Gamma \vdash \mathsf{let}\ \mathsf{mod}(-) \leftarrow M_0\ \mathsf{in}\ M_1 : B[\mathsf{id}.M_0]}$$

NB: the elimination rule forces $\Gamma.(\mu \mid A) \vdash \mathbf{p}.\mathsf{mod}(\mathbf{v}) : \Gamma.(\mathsf{id} \mid \langle \mu \mid A \rangle)$ to be anodyne.

Towards weak dependent adjunctions IV

There is a slight wrinkle in this story: what about $\langle \mu \mid \langle \mu \mid A \rangle \rangle$?

- Cannot apply our elimination rule to the inner modality.
- In fact, this type "stuck"; we can't manipulate $\Gamma.(\mu \mid \langle \mu \mid A \rangle)$

A solution: *crisp* induction principles. We'll ask that the following is anodyne:

$$\Gamma.(\nu \circ \mu \mid A) \vdash \mathbf{p}.\mathsf{mod}_{\mu}(\mathbf{v}) : \Gamma.(\nu \mid \langle \mu \mid A \rangle)$$

Towards weak dependent adjunctions V

We can render this crisp induction principle like the original elimination rule:

$$\frac{\Gamma.(\nu \mid \langle \mu \mid A \rangle) \vdash B \ \mathsf{type} \qquad \Gamma \vdash M_0 : \langle \mu \mid A \rangle \qquad \Gamma.(\nu \circ \mu \mid A) \vdash M_1 : B[\mathbf{p}.\mathsf{mod}(\mathbf{v})]}{\Gamma \vdash \mathsf{let}_{\nu}\ \mathsf{mod}(-) \leftarrow M_0 \ \mathsf{in}\ M_1 : B[\mathsf{id}.M_0]}$$

We refer to ν as the *framing* modality.

We require this rule to prove, e.g.:

$$\langle \nu \circ \mu \mid A \rangle \simeq \langle \nu \mid \langle \mu \mid A \rangle \rangle$$

Solution 4: MTT

Our next proposed theory is MTT [Gra+20]

- We can consider type theory with a series of *weak dependent adjunctions*
- Include all possible crisp induction principles for modalities
- The resulting theory admits normalization [Gra22]
- It can also be implemented [SGB22]
- There is a cubical variant, allowing for some homotopical models [Aag+22].
- It includes our two motivating examples.

Models of MTT

What does a model of MTT look like?

- Typically, models of MTT use DRAs.
- Automatically have all crisp induction principles

Theorem

Given a 2-functor $F : \mathcal{M} \longrightarrow \mathbf{Cat}$ assigning objects to categories with display maps and morphisms to right adjoints preserving display maps, F induces a model of MTT.

Various improvements are possible using e.g., the coherence result for pseudofunctors.

Corollary

Any model of DRATT is a model of MTT

From weak to strong dependent adjunctions

Under what assumptions is a weak dependent adjunction equivalent to a strong one?

- The answer isn't just "up to equality reflection":

$$\Gamma.(\mathsf{id} \mid \langle \mu \mid A \rangle).\{\mu\} \vdash \mathsf{?} : A[\mathbf{p}.\{\mu\}]$$

- Open question: is any closed term in DRATT definable in extensional MTT?

False for type theory with infinitary products. Similar to Davies and Pfenning [DP01].

Internal right adjoints

OK if our modality μ has a left adjoint in $\mathcal{M}$:

Theorem (Nuyts and Devriese [ND21])

In extensional MTT, internal right adjoints are dependent right adjoints.

The proof hinges on the fact that $-.\{-\}$ preserves adjoints and so

$$\Gamma.(\xi \mid A).\{\mu\} \cong \Gamma.\{\mu\}.(\nu \circ \xi \mid A[\cdots])$$

Allows us to "get behind" $-.\{\mu\}$ again.

FitchTT versus MTT

We now have two possible fixes to DRATT. The following relates them:

Corollary

The following are equivalent

1. *A nice model of extensional FitchTT whose left adjoints preserve terminal objects*
2. *A model of extensional MTT internalizing both the left and right adjoints.*

- If the left adjoint preserves terminal objects: go with MTT.
- If not: Cavallo [Cav21] has mixed the two. Shulman [Shu23] carries this further.

Solution 5: Left-division

We could hope that all modalities behave like these internal right adjoints.

- This is too strong, but it suffices to argue that $\mu \circ -$ is a right adjoint.
- A very strong but not vacuous requirement [ND18]
- The result: annotating everything in a context has a left adjoint in Cx.
- Unfolding this left adjoint, it corresponds to division of modalities.

Modalities are still weak DRAs but we have a stronger characterization of $-.\{\mu\}$.

From weak dependent right adjoints to dual-contexts

Where should we find an actual weak dependent adjunction?

- Slightly unlikely source: dual-contexts [PD01; dR15; Kav17; Shu18; Zwa19].
- The syntax of dual-contexts is based around a weak DRA
- The standard model gives a weak DRA which is not a DRA.

Solution 5: Dual-context type theory

We follow Zwanziger [Zwa19].

- Core idea: rather than a universal map $\Gamma \longrightarrow F(\Delta)$, provide a chosen one.
- Crucially, substitutions must also respect this chosen map!
- We represent this chosen map by splitting the context $\Delta; \Gamma$.

The end result is a new family of judgments over dual-contexts: $\Delta; \Gamma \vdash M : A$, etc.

Substitutions in dual-contexts

What sort of objects are substitutions between dual-contexts?

- The category of dual-contexts and substitutions is fibered over ordinary contexts.
- In particular, given $\Delta; \Gamma$ we can form $\Delta.A; \Gamma[\mathbf{p}^*]$.
- Each fiber has a terminal object $(\Delta; \mathbf{1})$
- Dual-contexts also have their own notion context extension $(\Delta; \Gamma.A)$

All of this can be crystallized in an algebraic style.

Modalities in dual-context type theory

We split the usual comonad into an adjunction:

$$\frac{\Delta; \mathbf{1} \vdash A}{\Delta \vdash \langle r \mid A \rangle} \qquad\qquad \frac{\Delta \vdash A}{\Delta; \Gamma \vdash \langle l \mid A \rangle}$$

In particular:

$$\frac{\Delta; \mathbf{1} \vdash A}{\Delta; \Gamma \vdash \Box A \triangleq \langle l \mid \langle r \mid A \rangle \rangle}$$

Modalities in dual-context type theory II

We have an adjunction between contexts and dual-contexts:

$$\Delta.\{r\} = \Delta;\mathbf{1} \qquad (\Delta;\Gamma).\{l\} = \Delta$$

We've assumed $\Delta;\mathbf{1}$ is terminal in the fiber over Δ.

$\Delta.A;\Gamma[\mathbf{p}]$ can now be given another name: $\Delta;\Gamma.(l \mid A)$

Modalities in dual-context type theory III

Summary of the remaining rules: $\langle l \mid - \rangle$ is a weak DRA and $\langle r \mid - \rangle$ is a DRA.

$$\frac{\Delta; \mathbf{1} \vdash M : A}{\Delta \vdash \mathsf{mod}_r(M) : \langle r \mid A \rangle} \qquad \frac{\Delta \vdash M : \langle r \mid A \rangle}{\Delta; \mathbf{1} \vdash \mathsf{unmod}_r(M) : A}$$

$$\frac{\Delta \vdash M : A}{\Delta; \Gamma \vdash \mathsf{mod}_l(M) : \langle l \mid A \rangle} \qquad \frac{\Delta; \Gamma \vdash M_0 : \langle l \mid A \rangle \qquad \Delta.A; \Gamma[\mathbf{p}] \vdash M_1 : B[\mathbf{p}^*.\mathsf{mod}_r(\mathbf{v})]}{\Delta; \Gamma \vdash \mathsf{let}\ \mathsf{mod}(-) \leftarrow M_0\ \mathsf{in}\ M_1 : B[\mathsf{id}.M_0]}$$

We *don't* include the crisp induction principles for $\langle l \mid - \rangle$.

The result is AdjTT [Zwa19].

Modalities in dual-context type theory IV

Q Why doesn't AdjTT suffer from the same issues as DRATT?

A $\langle r \mid - \rangle$ is an internal right adjoint, so $-.\{r\}$ is a (P)RA.

We can rephrase the elimination rule to something interderivable:

$$\frac{\Delta \vdash M : \langle r \mid A \rangle}{\Delta; \Gamma \vdash \mathsf{unmod}_r(M) : A}$$

MTT in dual-context type theory

Theorem

MTT with adjoint modalities but without crisp induction principles has an interpretation in AdjTT.

We can strengthen the result from Nuyts and Devriese [ND21] to apply:

Theorem

In the above situation, the right adjoint modality is a DRA without the η law.

The difference between this version of MTT and AdjTT comes from the fibration structure of dual-contexts over single contexts.

The standard model of AdjTT

The standard model of AdjTT [Zwa19] comes through gluing.

- Start with two finitely complete categories and a lex left adjoint $F : \mathcal{C} \longrightarrow \mathcal{D}$
- Dual-contexts are objects of $\mathbf{Gl}(F)$, single contexts are objects of $\mathcal{C}$
- Crucial point, display maps of $\mathbf{Gl}(F)$ are closed étale maps
- The interpretation of $\langle I \mid - \rangle$ is merely a weak DRA.

Combining our interpretation of MTT into AdjTT...

Corollary

MTT without crisp induction can be modeled by an arbitrary geometric morphism.

Corollary

MTT can be modeled by ... if the induced comonad is idempotent.

Stepping back

Thus far we've looked at 6 different modal type theories

- All except delayed substitutions use some form of (w)DRA for their modalities.
- As a consequence, MTT* can be interpreted into all of them.
- This doesn't imply MTT should always be used, of course!
- It does, however, place some limits on what modalities can be easily accommodated.

Still leaves certain questions frustratingly unanswered; not everything is a wDRA!

Shulman [Shu23] offers one approach by generalizing the gluing model for AdjTT.

Conclusions

Several open questions, several loose ends.

- Are interpretations of MTT into DRATT and idempotent AdjTT conservative?
- Plenty of metatheory left (cubical MTT, FitchTT, etc.)
- Can we include modalities which do not satisfy axiom K in this manner?
- Homotopical models? Some are alright (parametrized spectra over spaces)
- Left completely unexplored: substructural theories.

References

[Aag+22] Frederik Lerbjerg Aagaard, Magnus Baunsgaard Kristensen, Daniel Gratzer, and Lars Birkedal. *Unifying cubical and multimodal type theory*. 2022 (cit. on p. 39).

[Bd00] G. M. Bierman and V. C. V. de Paiva. "On an Intuitionistic Modal Logic". In: *Studia Logica* 65.3 (2000). DOI: 10.1023/A:1005291931660 (cit. on p. 14).

[BGM17] Patrick Bahr, Hans Bugge Grathwohl, and Rasmus Ejlers Møgelberg. "The clocks are ticking: No more delays!" In: *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. IEEE, 2017. DOI: 10.1109/LICS.2017.8005097. URL: http://www.itu.dk/people/mogel/papers/lics2017.pdf (cit. on pp. 24, 28).

[Bir+12] Lars Birkedal, Rasmus Møgelberg, Jan Schwinghammer, and Kristian Støvring. "First steps in synthetic guarded domain theory: step-indexing in the topos of trees". In: *Logical Methods in Computer Science* 8.4 (2012). Ed. by Patrick Baillot. DOI: 10.2168/LMCS-8(4:1)2012 (cit. on p. 7).

[Bir+20] Lars Birkedal, Ranald Clouston, Bassel Mannaa, Rasmus Ejlers Møgelberg, Andrew M. Pitts, and Bas Spitters. “Modal dependent type theory and dependent right adjoints”. In: *Mathematical Structures in Computer Science* 30.2 (2020), pp. 118–138. DOI: 10.1017/S0960129519000197. eprint: 1804.05236 (cit. on p. 20).

[Biz+16] Aleš Bizjak, Hans Bugge Grathwohl, Ranald Clouston, Rasmus E. Møgelberg, and Lars Birkedal. “Guarded Dependent Type Theory with Coinductive Types”. In: *Foundations of Software Science and Computation Structures*. Ed. by Bart Jacobs and Christof Löding. Springer Berlin Heidelberg, 2016, pp. 20–35 (cit. on p. 14).

[Cav21] Evan Cavallo. “Higher inductive types and internal parametricity for cubical type theory”. PhD thesis. Carnegie Mellon Univesity, 2021 (cit. on p. 43).

[DP01] Rowan Davies and Frank Pfenning. "A Modal Analysis of Staged Computation". In: *Journal of the ACM* 48.3 (May 2001), pp. 555–604 (cit. on p. 41).

[dR15] Valeria de Paiva and Eike Ritter. "Fibrational Modal Type Theory". In: *Proceedings of the Tenth Workshop on Logical and Semantic Frameworks, with Applications (LSFA 2015)*. 2015. DOI: 10.1016/j.entcs.2016.06.010 (cit. on p. 45).

[Gra+20] Daniel Gratzer, G.A. Kavvos, Andreas Nuyts, and Lars Birkedal. "Multimodal Dependent Type Theory". In: *Proceedings of the 35th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '20. ACM, 2020. DOI: 10.1145/3373718.3394736 (cit. on p. 39).

[Gra+22] Daniel Gratzer, Evan Cavallo, G. A. Kavvos, Adrien Guatto, and Lars Birkedal. "Modalities and Parametric Adjoints". In: *ACM Trans. Comput. Logic* 23.3 (Apr. 2022). ISSN: 1529-3785. DOI: 10.1145/3514241. URL: https://doi.org/10.1145/3514241 (cit. on p. 30).

[Gra22] Daniel Gratzer. "Normalization for Multimodal Type Theory". In: *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '22. Haifa, Israel: Association for Computing Machinery, 2022. ISBN: 9781450393515. DOI: 10.1145/3531130.3532398. URL: https://doi.org/10.1145/3531130.3532398 (cit. on p. 39).

[GSB19] Daniel Gratzer, Jonathan Sterling, and Lars Birkedal. “Implementing a Modal Dependent Type Theory”. In: *Proc. ACM Program. Lang.* 3 (ICFP 2019). DOI: 10.1145/3341711 (cit. on p. 24).

[HP23] Jason Z. S. Hu and Brigitte Pientka. “A Categorical Normalization Proof for the Modal Lambda-Calculus”. In: *Electronic Notes in Theoretical Informatics and Computer Science* Volume 1 - Proceedings of MFPS XXXVIII (Feb. 2023). DOI: 10.46298/entics.10360 (cit. on p. 24).

[Kav17] G. A. Kavvos. “Dual-context calculi for modal logic”. In: *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*. 2017, pp. 1–12. DOI: 10.1109/LICS.2017.8005089. arXiv: 1602.04860 (cit. on p. 45).

[Lic+18] Daniel R. Licata, Ian Orton, Andrew M. Pitts, and Bas Spitters. "Internal Universes in Models of Homotopy Type Theory". In: *3rd International Conference on Formal Structures for Computation and Deduction (FSCD 2018)*. Ed. by H. Kirchner. Leibniz International Proceedings in Informatics (LIPIcs). Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018, 22:1–22:17. DOI: 10.4230/LIPIcs.FSCD.2018.22. eprint: 1801.07664 (cit. on p. 6).

[LW15] Peter Lefanu Lumsdaine and Michael A. Warren. "The local universes model, an overlooked coherence construction for dependent type theories". In: *ACM Transactions on Computational Logic* 16.3 (2015). DOI: 10.1145/2754931 (cit. on p. 11).

[ND18] Andreas Nuyts and Dominique Devriese. “Degrees of Relatedness: A Unified Framework for Parametricity, Irrelevance, Ad Hoc Polymorphism, Intersections, Unions and Algebra in Dependent Type Theory”. In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS ’18. ACM, 2018. DOI: 10.1145/3209108.3209119 (cit. on p. 44).

[ND21] Andreas Nuyts and Dominique Devriese. *Transpension: The Right Adjoint to the Pi-type*. 2021 (cit. on pp. 42, 52).

[OP18] Ian Orton and Andrew M. Pitts. “Axioms for Modelling Cubical Type Theory in a Topos”. In: *Logical Methods in Computer Science* 14.4 (2018). DOI: 10.23638/LMCS-14(4:23)2018. arXiv: 1712.04864 (cit. on p. 6).

[PD01] Frank Pfenning and Rowan Davies. "A Judgmental Reconstruction of Modal Logic". In: *Mathematical Structures in Computer Science* 11.4 (2001), pp. 511–540. DOI: 10.1017/S0960129501003322. URL: http://www.cs.cmu.edu/~fp/papers/mscs00.pdf (cit. on p. 45).

[SGB22] Philipp Stassen, Daniel Gratzer, and Lars Birkedal. *mitten: a flexible multimodal proof assistant*. 2022 (cit. on p. 39).

[Shu18] Michael Shulman. "Brouwer's fixed-point theorem in real-cohesive homotopy type theory". In: *Mathematical Structures in Computer Science* 28.6 (2018), pp. 856–941. DOI: 10.1017/S0960129517000147. URL: https://doi.org/10.1017/S0960129517000147 (cit. on pp. 6, 45).

[Shu23] Michael Shulman. *Semantics of multimodal adjoint type theory*. 2023. URL: https://arxiv.org/abs/2303.02572 (cit. on pp. 43, 54).

[Zwa19] Colin Zwanziger. “Natural Model Semantics for Comonadic and Adjoint Type Theory: Extended Abstract”. In: *Preproceedings of Applied Category Theory Conference 2019*. 2019 (cit. on pp. 45, 46, 50, 53).