Normalization for multimodal type theory

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Contributions

We present a normalization proof for MTT, a multimodal dependent type theory.

1. Conversion in MTT is decidable iff the collection of modalities is decidable
2. Type-checking MTT is decidable under the same conditions

The takeaway: MTT can be implemented for every sensible mode theory.

An inadequate summary of MTT

We must begin by recalling some of MTT, a multimodal type theory [Gra+20]

- Start with a mode theory $\mathcal{M}$
- Add a distinct copies of MLTT for each $m : \mathcal{M}$
- Add a modal type for $\mu : n \longrightarrow m$

$$\frac{\Gamma.\{\mu\} \vdash M : A \mathbin{@} n}{\Gamma \vdash \mathsf{mod}_\mu(M) : \langle \mu \mid A \rangle \mathbin{@} m}$$

Normalization for MTT

What makes normalization challenging?

- Surprisingly, it's not really the modalities!
- The real challenge is the amount of data in play.

Can quickly get buried beneath all the data.

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The solution: gluing! [AHS95; Str98; Alt+01; Fio02; AK16; Shu15; KHS19; Coq19]

Rather than constructing an algorithm, build a model and deduce normalization.

Normalization by Gluing for MTT

It's still quite challenging to construct a gluing model directly.

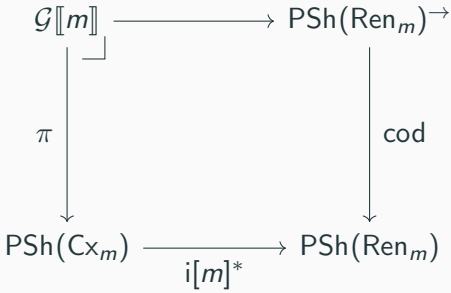
A few minor adjustments:

- A collection of CwFs becomes a collection of LCCCs.
- Morphisms only preserve some structures up to isomorphism.

We can still obtain normalization for the initial *strict* model of MTT...
but the more flexible structures let us work more abstractly.

The gluing category for mode m

For a mode m, we have a category of contexts and of renamings: $\mathsf{i}[m] : \mathsf{Ren}_m \longrightarrow \mathsf{Cx}_m$



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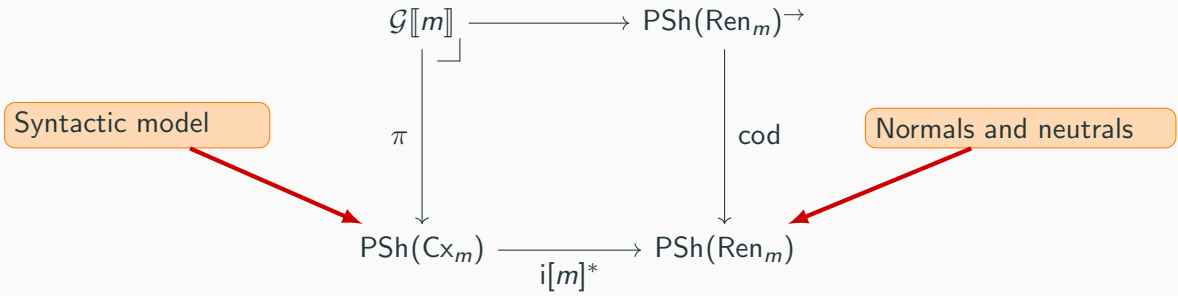

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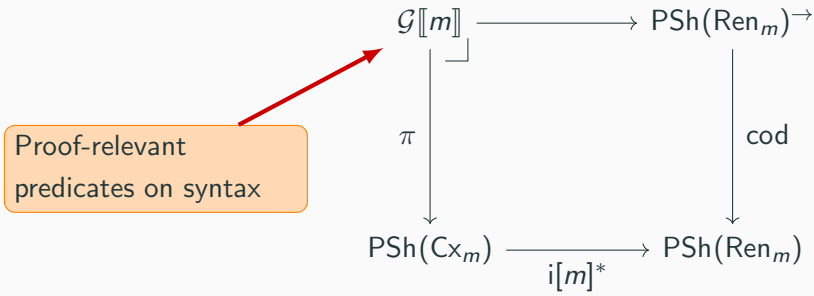

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Constructing a model in $\mathcal{G}[\![-]\!]$

We now must interpret types, terms, etc. into $\mathcal{G}[\![-]\!]$. To do this, we use MTT.

Theorem
There is a model of extensional MTT in $\mathcal{G}[\![-]\!]$; modalities precompose with $-.\{\mu\}$.

Not the normalization model.

An internal language for the *network* of categories $\mathcal{G}[\![m]\!]$.

Building the normalization model

We extend this internal language and define the normalization model internally.

- Interpretations of terms/types and the reify/reflect are done internally
- *Synthetic Tait computability* for a modal setting [SH20; SG20; SA21; Ste21]

The complicated bookkeeping is now handled by the internal language!

Theorem
There exists a model of MTT in $\mathcal{G}[\![-]\!]$ *lying over the syntactic model in* $\mathsf{PSh}(\mathsf{Cx}_{-})$.

Proof summary

In summary

- We define a normalization algorithm by building a particular model of MTT
- We crucially leverage MTT as the internal language of the gluing categories
- This approach extends modern gluing techniques to multimodal theories

In conclusion: MTT is implementable for a wide class of mode theories.

https://arxiv.org/abs/2106.01414