Modalities and Parametric Adjoints

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Modalities in type theory

Modal type theory is full of compromises:

Goal:	Add a modality to MLTT
Problem:	The modality is poorly behaved
Compromise:	Leverage another property to make the system workable

Example

The dual-context calculi require functorial modalities preserving finite limits.

One consequence is the massive proliferation of modal type theories.

Taming the chaos

Nowhere near a "solution" to this problem, but different schools are emerging.

Particularly interesting to us: Fitch style modal type theories.

- Modalities are right adjoints
- Resultant theories have natural syntax (no `let`-binding hell!)
- Strong β and η rules

Fitch-style modalities and DRA

We add a left adjoint operator to the context and define rules based on transposition.

Operators:

$$\frac{\Gamma\ \mathsf{cx}}{\Gamma.\text{🔒}\ \mathsf{cx}} \qquad \frac{\Gamma.\text{🔒} \vdash M : A}{\Gamma \vdash \mathsf{mod}(M) : \Box A} \qquad \frac{\Gamma \vdash M : \Box A}{\Gamma.\text{🔒} \vdash \mathsf{unmod}(M) : A}$$

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Equations:

$$\Gamma.\text{🔒} \vdash \mathsf{unmod}(\mathsf{mod}(M)) = M : A \qquad \Gamma \vdash \mathsf{mod}(\mathsf{unmod}(M)) = M : \Box A$$

$$\Gamma \vdash \mathsf{mod}(M)[\delta] = \mathsf{mod}(M[\delta.\text{🔒}]) : \Box(A[\delta.\text{🔒}])$$

What about substitutions applied to $\mathsf{unmod}(-)$?

The thorny bit

In fact, there is no obvious way to push substitutions past unmod$(-)$:

- Suppose $\Delta \vdash M : \Box A$ so $\Delta.\text{🔒} \vdash \text{unmod}(M) : A$
- Suppose $\delta : \Gamma \longrightarrow \Delta.\text{🔒}$
- Want M_0 such that $\Gamma \vdash \text{unmod}(M_0) : A[\delta]$

Clearly nonsense if $\Gamma \neq \Gamma_0.\text{🔒}$ or $\delta \neq \delta_0.\text{🔒}$.

With DRA, only exceptions are weakenings; we can adapt unmod$(-)$ in an ad hoc way.

Contributions

We recognize the underlying structure necessary for this ad hoc adaption of unmod$(-)$:

Big Idea

$-.🔒$ must be a parametric right adjoint (PRA).

From this observation, we generalize to FitchTT with multiple interacting modalities.

- Prove that $-.🔒$ is a PRA in all prior Fitch style type theories
- Prove FitchTT is conservative over DRA
- Show specialized calculi (CloTT, ParamTT) embed FitchTT

At a high level: we de-mystify the syntax of Fitch style type theories.

A special case

The remainder of this talk is focused on explaining the new elimination rule for $\Box A$.

Helpful to start with a concrete example:

- Fix a closed type $\Gamma \vdash \mathfrak{C}$ type
- Define $\Box A = \mathfrak{C} \to A$
- Define $\Gamma.🔒 = \Gamma.\mathfrak{C}$

The $\mathsf{mod}(-)$ rule is recognizable as λ.

Question

What about $\mathsf{unmod}(-)$?

$\mathsf{unmod}(-)$ for dependent products

$$\frac{\Gamma \vdash M : \mathfrak{C} \to A}{\Gamma.\mathfrak{C} \vdash \mathsf{unmod}(M) : A}$$

Valid, same substitution issues but equivalent to application:

$$\frac{\Gamma \vdash M : \mathfrak{C} \to A \qquad r : \Gamma \longrightarrow 1.\mathfrak{C}}{\Gamma \vdash M \mathbin{@} r : A[\mathsf{id}.\mathsf{v}_0[r]]}$$

Question

Why are these equivalent?

A substitution $\Delta \longrightarrow \Gamma.\mathfrak{C}$ is precisely determined by $\Delta \longrightarrow \Gamma$ and $\Delta \longrightarrow 1.\mathfrak{C}$

Enter PRAs

Let's axiomatize this behavior.

Definition

A PRA is a functor $G\colon \mathcal{C} \longrightarrow \mathcal{D}$ such that $G\colon \mathcal{C}/1 \longrightarrow \mathcal{D}/G(1)$ is a right adjoint.

Theorem

$G(\Gamma) = \Gamma.\mathfrak{C}$ is a PRA whose left adjoint sends $\Delta \longrightarrow 1.\mathfrak{C}$ to Δ.

The new elimination rule

Let's return to the application rule. Write L for the projection $\mathcal{C}/\mathfrak{C} \longrightarrow \mathcal{C}$.

$$\frac{L(\Gamma, r) \vdash M : \mathfrak{C} \to A \qquad r : \Gamma \longrightarrow 1.\mathfrak{C}}{\Gamma \vdash M \mathbin{@} r : A}$$

This rule is now interderivable with $\mathsf{unmod}(-)$ by the purely formal properties of PRAs.

Big Idea

Let's forget $\Gamma.\text{🔒}$ is $\Gamma.\mathfrak{C}$ and just demand $-.\text{🔒}$ is a PRA

FitchTT **with one modality**

We can summarize:

$$\frac{\Gamma\ \mathsf{cx}}{\Gamma.🔒\ \mathsf{cx}} \qquad \frac{\Gamma\ \mathsf{cx} \qquad r : \Gamma \longrightarrow 1.🔒}{\Gamma/(r : 🔒)\ \mathsf{cx}}$$

$$\frac{\Gamma.🔒 \vdash M : A}{\Gamma \vdash \mathsf{mod}(M) : \Box A} \qquad \frac{\Gamma/(r : 🔒) \vdash M : \Box A \qquad r : \Gamma \longrightarrow 1.🔒}{\Gamma \vdash M \mathbin{@} r : A[\eta[r]]}$$

Not shown (properly): substitutions making $-.🔒$ a PRA with left adjoint $-/(- : 🔒)$

It's now easy to show this type theory enjoys substitution.

Multiple modalities

Remarkably, almost no effort is required to scale up

- Fix a 2-category of modes, modalities, and transformations.
- Each modality gives rise to a $-.🔒$, written $-.\{\mu\}$.
- Each $-.\{\mu\}$ is a parametric right adjoint with left adjoint $-/(- : \mu)$

The rules for each modality are identical to the single modality case!

We call the resulting system FitchTT.

Are PRAs inevitable?

Lots of Fitch style type theories in nature...

Question

Why haven't PRAs been mentioned before?

- In fact, in prior type theories $-.\text{🔒}$ was a PRA... in the syntactic model.
- Used to prove substitution admissible.
- However, it meant these languages were difficult to use as an internal language.

Even if not explicit, PRAs are central to all prior Fitch-style type theories!

Theorem

FitchTT *with a single modality conservatively extends* DRA

Are PRAs convenient?

In fact, many specialized calculi have equivalents of $-/(- : \mu)$

- Nominal and parametric type theory and name/dimension restriction
- Substructurality of ticks in clocked type theory

We can recover the specialized syntax through FitchTT!

Conclusions

In summary

- We recognize the centrality of PRAs in Fitch style type theories.
- Introduce FitchTT, a general multimodal Fitch style type theory.
- Prove FitchTT is conservative over DRA
- Show specialized calculi (CloTT, ParamTT) embed FitchTT

`jozefg.github.io/papers/modalities-and-parametric-adjoints.pdf`