Normalization by Evaluation for Modal Dependent Type Theory

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Contents

1 Modal Dependent Type Theory

Here, we treat the syntax of $\mathsf{MLTT}_{\text{🔒}}$, a modal dependent type theory with typed definitional equality and a predicative hierarchy of universes.

1.1 The syntax of $\mathsf{MLTT}_{\text{🔒}}$

We represent the syntax of TT abstractly using De Bruijn indices and explicit substitutions [Dyb96; Gra13]. By convention, we use a distinguished color for syntactic objects (as opposed to the semantic objects that we will introduce in later chapters).

$$
\begin{array}{llcl}
\textit{(contexts)} & \Gamma, \Delta & ::= & \cdot \mid \Gamma.A \mid \Gamma.\text{🔒} \\
\textit{(types)} & A, B, T & ::= & t \mid \mathsf{nat} \mid \mathsf{U}_i \mid \Pi(A, B) \mid \Sigma(A, B) \mid \square A \mid \mathsf{Id}(A, t, t) \\
\textit{(terms)} & s, t & ::= & A \mid \mathsf{var}_n \mid \lambda(t) \mid t(t) \mid \langle t, t \rangle \mid \mathsf{fst}(t) \mid \mathsf{snd}(t) \mid [t]_{\text{🔒}} \mid [t]_{\text{🔓}} \mid \\
 & & & \mathsf{refl}(t) \mid \mathsf{J}(C, t, t) \mid \mathsf{zero} \mid \mathsf{succ}(t) \mid \mathsf{natrec}(A, t, t, t) \mid t[\delta] \\
\textit{(subst.)} & \gamma, \delta & ::= & \mathsf{id} \mid \delta.t \mid \delta \circ \delta \mid \mathsf{p}^n \mid \cdot
\end{array}
$$

We now turn to the typing rules for this calculus. We write $\Gamma^{\text{🔓}}$ for the operation which removes all locks from a context. We write $\Gamma \triangleright_{\text{🔒}} \Gamma'$ to mean that Γ' is a version of Γ with locks added.

$$\boxed{\Gamma_0 \triangleright_{\text{🔒}} \Gamma_1}$$

$$
\frac{\Gamma \; \mathit{ctx}}{\Gamma \triangleright_{\text{🔒}} \Gamma}
\qquad
\frac{\Gamma_0 \triangleright_{\text{🔒}} \Gamma_1 \qquad \Gamma_1 \triangleright_{\text{🔒}} \Gamma_2}{\Gamma_0 \triangleright_{\text{🔒}} \Gamma_2}
\qquad
\frac{\Gamma_0 \vdash A \; \mathit{type} \qquad \Gamma_1 \vdash A \; \mathit{type} \qquad \Gamma_0 \triangleright_{\text{🔒}} \Gamma_1}{\Gamma_0.A \triangleright_{\text{🔒}} \Gamma_1.A}
\qquad
\frac{\Gamma_0 \triangleright_{\text{🔒}} \Gamma_1}{\Gamma_0.\text{🔒} \triangleright_{\text{🔒}} \Gamma_1.\text{🔒}}
$$

$$
\frac{\Gamma.\text{🔒}.T \; \mathit{ctx}}{\Gamma.\text{🔒}.T \triangleright_{\text{🔒}} \Gamma.T.\text{🔒}}
\qquad
\frac{\Gamma \; \mathit{ctx}}{\Gamma \triangleright_{\text{🔒}} \Gamma.\text{🔒}}
\qquad
\frac{\Gamma \; \mathit{ctx}}{\Gamma.\text{🔒}.\text{🔒} \triangleright_{\text{🔒}} \Gamma.\text{🔒}}
$$

$$\boxed{\Gamma \; \mathit{ctx}}$$

$$
\frac{}{\cdot \; \mathit{ctx}}
\qquad
\frac{\Gamma \; \mathit{ctx}}{\Gamma.\text{🔒} \; \mathit{ctx}}
\qquad
\frac{\Gamma \; \mathit{ctx} \qquad \Gamma \vdash T \; \mathit{type}}{\Gamma.T \; \mathit{ctx}}
$$

$$\boxed{\Gamma \vdash T \; \mathit{type}}$$

$$
\frac{\Gamma \; \mathit{ctx}}{\Gamma \vdash \mathsf{U}_i \; \mathit{type} \qquad \Gamma \vdash \mathsf{nat} \; \mathit{type}}
\qquad
\frac{\Gamma.\text{🔒} \vdash T \; \mathit{type}}{\Gamma \vdash \square T \; \mathit{type}}
\qquad
\frac{\Gamma \vdash T_0 \; \mathit{type} \qquad \Gamma.T_0 \vdash T_1 \; \mathit{type}}{\Gamma \vdash \Pi(T_0, T_1) \; \mathit{type} \qquad \Gamma \vdash \Sigma(T_0, T_1) \; \mathit{type}}
$$

$$
\frac{\Gamma \vdash T : \mathsf{U}_i}{\Gamma \vdash T \; \mathit{type}}
\qquad
\frac{\Gamma \vdash T \; \mathit{type} \qquad \Gamma \vdash t_0, t_1 : T}{\Gamma \vdash \mathsf{Id}(T, t_0, t_1) \; \mathit{type}}
\qquad
\frac{\Gamma \vdash \delta : \Delta \qquad \Delta \vdash T \; \mathit{type}}{\Gamma \vdash T[\delta] \; \mathit{type}}
$$

$$\boxed{\Gamma \vdash t : T}$$

$$\frac{\Gamma_0.T.\Gamma_1\ \mathit{ctx} \qquad \text{🔒} \notin \Gamma_1 \qquad k = \|\Gamma_1\|}{\Gamma_0.T.\Gamma_1 \vdash \mathsf{var}_k : T[\mathsf{p}^{k+1}]} \qquad \frac{\Gamma \vdash A\ \mathit{type} \qquad \Gamma.A \vdash t : B}{\Gamma \vdash \lambda(t) : \Pi(A, B)}$$

$$\frac{\Gamma \vdash t : \Pi(A, B) \qquad \Gamma \vdash u : A \qquad \Gamma.A \vdash B\ \mathit{type}}{\Gamma \vdash t(u) : B[\mathsf{id}.u]} \qquad \frac{\Gamma \vdash A : \mathsf{U}_i \qquad \Gamma.A \vdash B : \mathsf{U}_i}{\Gamma \vdash \Pi(A, B) : \mathsf{U}_i}$$

$$\frac{\Gamma \vdash t_0 : A \qquad \Gamma.A \vdash B\ \mathit{type} \qquad \Gamma \vdash t_1 : B[\mathsf{id}.t_0]}{\Gamma \vdash \langle t_0, t_1 \rangle : \Sigma(A, B)} \qquad \frac{\Gamma \vdash t : \Sigma(A, B) \qquad \Gamma \vdash A\ \mathit{type}}{\Gamma \vdash \mathsf{fst}(t) : A}$$

$$\frac{\Gamma \vdash t : \Sigma(A, B) \qquad \Gamma \vdash A\ \mathit{type} \qquad \Gamma.A \vdash B\ \mathit{type}}{\Gamma \vdash \mathsf{snd}(t) : B[\mathsf{id}.(\mathsf{fst}(t))]} \qquad \frac{\Gamma \vdash A : \mathsf{U}_i \qquad \Gamma.A \vdash B : \mathsf{U}_i}{\Gamma \vdash \Sigma(A, B) : \mathsf{U}_i} \qquad \frac{\Gamma\ \mathit{ctx}}{\Gamma \vdash \mathsf{zero} : \mathsf{nat}}$$

$$\frac{\Gamma \vdash t : \mathsf{nat}}{\Gamma \vdash \mathsf{succ}(t) : \mathsf{nat}}$$

$$\frac{\Gamma.\mathsf{nat} \vdash A\ \mathit{type} \qquad \Gamma \vdash t_n : \mathsf{nat} \qquad \Gamma \vdash t_z : A[\mathsf{id}.\mathsf{zero}] \qquad \Gamma.\mathsf{nat}.A \vdash t_s : A[\mathsf{p}^2.\mathsf{succ}(\mathsf{var}_1)]}{\Gamma \vdash \mathsf{natrec}(A, t_n, t_z, t_s) : A[\mathsf{id}.t_n]}$$

$$\frac{\Gamma\ \mathit{ctx}}{\Gamma \vdash \mathsf{nat} : \mathsf{U}_i} \qquad \frac{\Gamma \vdash T : \mathsf{U}_i \qquad \Gamma \vdash t_0, t_1 : T}{\Gamma \vdash \mathsf{Id}(T, t_0, t_1) : \mathsf{U}_i} \qquad \frac{\Gamma \vdash T\ \mathit{type} \qquad \Gamma \vdash t : T}{\Gamma \vdash \mathsf{refl}(t) : \mathsf{Id}(T, t, t)}$$

$$\frac{\begin{array}{c}\Gamma \vdash T\ \mathit{type} \qquad \Gamma \vdash u_0, u_1 : T \qquad \Gamma.T.T[\mathsf{p}^1].\mathsf{Id}(T[\mathsf{p}^2], \mathsf{var}_1, \mathsf{var}_0) \vdash C\ \mathit{type} \\ \Gamma.T \vdash t_0 : C[\mathsf{id}.\mathsf{var}_0.\mathsf{var}_0.\mathsf{refl}(\mathsf{var}_0)] \qquad \Gamma \vdash t_1 : \mathsf{Id}(T, u_0, u_1)\end{array}}{\Gamma \vdash \mathsf{J}(C, t_0, t_1) : C[\mathsf{id}.u_0.u_1.t_1]} \qquad \frac{\Gamma.\text{🔒} \vdash t : A}{\Gamma \vdash [t]_{\text{🔒}} : \Box A}$$

$$\frac{\Gamma \vdash A\ \mathit{type} \qquad \Gamma^{\text{🔓}} \vdash t : \Box A}{\Gamma \vdash [t]_{\text{🔓}} : A} \qquad \frac{\Gamma.\text{🔒} \vdash A : \mathsf{U}_i}{\Gamma \vdash \Box A : \mathsf{U}_i} \qquad \frac{\Gamma\ \mathit{ctx}}{\Gamma \vdash \mathsf{U}_i : \mathsf{U}_{i+1}} \qquad \frac{\Gamma \vdash A : \mathsf{U}_i}{\Gamma \vdash A : \mathsf{U}_{i+1}}$$

$$\frac{\Gamma \vdash \delta : \Delta \qquad \Delta \vdash t : A}{\Gamma \vdash t[\delta] : A[\delta]} \qquad \frac{\Gamma \vdash A = B\ \mathit{type} \qquad \Gamma \vdash t : A}{\Gamma \vdash t : B}$$

$$\boxed{\Gamma \vdash \delta : \Delta}$$

$$\frac{\Gamma\ \mathit{ctx} \qquad \Delta\ \mathit{ctx} \qquad \cdot \rhd_{\text{🔒}} \Delta}{\Gamma \vdash \cdot : \Delta} \qquad \frac{\Gamma_1\ \mathit{ctx} \qquad \Gamma_2\ \mathit{ctx} \qquad \Gamma_1 \rhd_{\text{🔒}} \Gamma_2}{\Gamma_1 \vdash \mathsf{id} : \Gamma_2}$$

$$\frac{\Delta \vdash T\ \mathit{type} \qquad \Gamma \vdash \delta : \Delta \qquad \Gamma \vdash t : T[\delta]}{\Gamma \vdash \delta.t : \Delta.T} \qquad \frac{\Gamma_0 \vdash \gamma_1 : \Gamma_1 \qquad \Gamma_1 \vdash \gamma_2 : \Gamma_2}{\Gamma_0 \vdash \gamma_2 \circ \gamma_1 : \Gamma_2} \qquad \frac{\Gamma_0\ \mathit{ctx} \qquad \Gamma_0^{\text{🔓}} \vdash \gamma_1 : \Gamma_1}{\Gamma_0 \vdash \gamma_1 : \Gamma_1.\text{🔒}}$$

$$\frac{\Gamma_0.\Gamma_1\ \mathit{ctx} \qquad \Gamma_0'\ \mathit{ctx} \qquad \Gamma_0 \rhd_{\text{🔒}} \Gamma_0' \qquad k = \|\Gamma_1\| \qquad \text{🔒} \notin \Gamma_1}{\Gamma_0.\Gamma_1 \vdash \mathsf{p}^k : \Gamma_0'}$$

We omit most of the rules for definitional equality, which are standard, presenting only those which pertain to the new type connectives. We have equipped both depenent function and dependent pair types with the appropriate η rules. The rules the $\Box$ connective are specified below.

$$\frac{\Gamma \vdash T\ \mathit{type} \qquad \Delta \vdash \mathsf{id} : \Gamma}{\Delta \vdash T[\mathsf{id}] = T\ \mathit{type}} \qquad \frac{\Gamma_0 \vdash \gamma_1 : \Gamma_1 \qquad \Gamma_1 \vdash \gamma_2 : \Gamma_2 \qquad \Gamma_2 \vdash T\ \mathit{type}}{\Gamma_0 \vdash T[\gamma_2][\gamma_1] = T[\gamma_2 \circ \gamma_1]\ \mathit{type}} \qquad \frac{\Gamma \vdash t : T \qquad \Delta \vdash \mathsf{id} : \Gamma}{\Delta \vdash t[\mathsf{id}] = t : T}$$

$$\frac{\Gamma_0 \vdash \gamma_1 : \Gamma_1 \qquad \Gamma_1 \vdash \gamma_2 : \Gamma_2 \qquad \Gamma_2 \vdash t : T}{\Gamma_0 \vdash t[\gamma_2][\gamma_1] = t[\gamma_2 \circ \gamma_1] : T[\gamma_2 \circ \gamma_1]} \qquad \frac{\Gamma.\text{🔒} \vdash A = B\ \mathit{type}}{\Gamma \vdash \Box A = \Box B\ \mathit{type}} \qquad \frac{\Gamma.\text{🔒} \vdash A = B : \mathsf{U}_i}{\Gamma \vdash \Box A = \Box B : \mathsf{U}_i}$$

$$\frac{\Gamma.\text{🔒} \vdash t_0 = t_1 : A}{\Gamma \vdash [t_0]_{\text{🔒}} = [t_1]_{\text{🔒}} : \Box A} \qquad \frac{\Gamma \vdash A\ \mathit{type} \qquad \Gamma^{\text{🔓}} \vdash t_0 = t_1 : \Box A}{\Gamma \vdash [t_0]_{\text{🔓}} = [t_1]_{\text{🔓}} : A} \qquad \frac{\Gamma^{\text{🔓}}.\text{🔒} \vdash t : A}{\Gamma \vdash [[t]_{\text{🔒}}]_{\text{🔓}} = t : A}$$

$$\frac{\Gamma \vdash t : \Box A}{\Gamma \vdash [[t]_{\text{🔓}}]_{\text{🔒}} = t : \Box A} \qquad \frac{\Gamma \vdash \delta : \Delta \qquad \Delta.\text{🔒} \vdash A\ \mathit{type}}{\Gamma \vdash (\Box A)[\delta] = \Box(A[\delta])\ \mathit{type}} \qquad \frac{\Gamma \vdash \delta : \Delta \qquad \Delta.\text{🔒} \vdash t : T}{\Gamma \vdash [t]_{\text{🔒}}[\delta] = [t[\delta]]_{\text{🔒}} : (\Box T)[\delta]}$$

$$\frac{\Gamma \vdash \delta : \Delta \qquad \Delta^{\text{🔓}} \vdash t : \Box T}{\Gamma \vdash [t]_{\text{🔓}}[\delta] = [t[\delta]]_{\text{🔓}} : T[\delta]}$$

The rules for equality of substitution are largely standard, but presented in a more general way in order to properly mediate the presence of 🔒.

$$\frac{\Gamma_0 \vdash \mathsf{p}^1.\mathsf{var}_0 : \Gamma_1 \qquad \Gamma_0 \vdash \mathsf{id} : \Gamma_1}{\Gamma_0 \vdash \mathsf{p}^1.\mathsf{var}_0 = \mathsf{id} : \Gamma_1} \qquad \frac{\Gamma_0 \vdash \gamma_1 : \Gamma_1 \qquad \Gamma_1 \vdash \gamma_2 : \Gamma_2 \qquad \Gamma_2 \vdash \gamma_3 : \Gamma_3}{\Gamma_0 \vdash \gamma_3 \circ (\gamma_2 \circ \gamma_1) = (\gamma_3 \circ \gamma_2) \circ \gamma_1 : \Gamma_3}$$

$$\frac{\Gamma_0 \vdash \gamma_1 : \Gamma_1 \qquad \Gamma_1 \vdash \mathsf{id} : \Gamma_2}{\Gamma_0 \vdash \mathsf{id} \circ \gamma_1 = \gamma_1 : \Gamma_2} \qquad \frac{\Gamma_0 \vdash \mathsf{id} : \Gamma_1 \qquad \Gamma_1 \vdash \gamma_2 : \Gamma_2}{\Gamma_0 \vdash \gamma_2 \circ \mathsf{id} = \gamma_2 : \Gamma_2} \qquad \frac{\Gamma_1 \vdash \gamma_2 : \Gamma_2 \qquad \Gamma_2 \vdash \gamma.t : \Gamma_3}{\Gamma_1 \vdash (\gamma.t) \circ \gamma_2 = (\gamma \circ \gamma_2).(t[\gamma_2]) : \Gamma_3}$$

$$\frac{\Gamma_0 \vdash \mathsf{p}^{n+1} : \Gamma_1}{\Gamma_0 \vdash \mathsf{p}^{n+1} = \mathsf{p}^n \circ \mathsf{p}^1 : \Gamma_1} \qquad \frac{\Gamma_0 \vdash \gamma.t : \Gamma_1 \qquad \Gamma_1 \vdash \mathsf{p}^1 : \Gamma_2}{\Gamma_0 \vdash \mathsf{p}^1 \circ (\gamma.t) = \gamma : \Gamma_2}$$

1.2 Admissible rules

In this section, we prove a number of critical admissible rules which will be exploited throughout the rest of this report. In what follows we use $\mathcal{J}$ to stand for any of the judgments of $\mathsf{MLTT}_{\text{🔒}}$.

Proposition 1.2.1 (Lock-variable exchange). *Supposing that* $\Gamma.\text{🔒} \vdash T$ *type holds if* $\Gamma_0.T.\text{🔒}.\Gamma_1 \vdash \mathcal{J}$ *then* $\Gamma_0.\text{🔒}.T.\Gamma_1 \vdash \mathcal{J}$.

Proof. Proven in Theorem 1.2.7. □

Proposition 1.2.2 (Lock strengthening). *If* $\Gamma_0.\text{🔒}.\Gamma_1 \vdash \mathcal{J}$ *then* $\Gamma_0.\Gamma_1 \vdash \mathcal{J}$.

Proof. Proven in Theorem 1.2.4. □

Proposition 1.2.3 (Presuppositions).

1. *If* $\Gamma \vdash T$ *type then* Γ *ctx.*
2. *If* $\Gamma \vdash t : T$ *then* $\Gamma \vdash T$ *type.*
3. *If* $\Gamma_0 \vdash \delta : \Gamma_1$ *then* Γ_i *ctx.*
4. *If* $\Gamma \vdash T_0 = T_1$ *type then* $\Gamma \vdash T_i$ *type.*

5. *If* $\Gamma \vdash t_0 = t_1 : T$ *then* $\Gamma \vdash t_i : T$.

6. *If* $\Gamma \vdash \delta_0 = \delta_1 : \Delta$ *then* $\Gamma \vdash \delta_i : \Delta$.

Proof. Proven in Theorem 1.2.16. □

Theorem 1.2.4 (Lock Strengthening)**.**

1. *If* $\Gamma_0.\text{🔒}.\Gamma_1$ *ctx then* $\Gamma_0.\Gamma_1$ *ctx.*
2. *If* $\Gamma_0.\text{🔒}.\Gamma_1 \vdash T$ *type then* $\Gamma_0.\Gamma_1 \vdash T$ *type.*
3. *If* $\Gamma_0.\text{🔒}.\Gamma_1 \vdash T_0 = T_1$ *type then* $\Gamma_0.\Gamma_1 \vdash T_0 = T_1$ *type.*
4. *If* $\Gamma_0.\text{🔒}.\Gamma_1 \vdash t : T$ *then* $\Gamma_0.\Gamma_1 \vdash t : T$.
5. *If* $\Gamma_0.\text{🔒}.\Gamma_1 \vdash t_0 = t_1 : T$ *then* $\Gamma_0.\Gamma_1 \vdash t_0 = t_1 : T$.
6. *If* $\Gamma_0.\text{🔒}.\Gamma_1 \vdash \delta : \Delta$ *then* $\Gamma_0.\Gamma_1 \vdash \delta : \Delta$.
7. *If* $\Gamma_0.\text{🔒}.\Gamma_1 \vdash \delta_0 = \delta_1 : \Delta$ *then* $\Gamma_0.\Gamma_1 \vdash \delta_0 = \delta_1 : \Delta$.

Proof. These facts must be proved mutually as these judgments are all mutual. They are all proven by induction on the derivation; for brevity, we present only a few representative cases involving locks.

1. If $\Gamma_0.\text{🔒}.\Gamma_1$ *ctx* then $\Gamma_0.\Gamma_1$ *ctx*.

   Case.
   $$\frac{\Gamma_0.\text{🔒}.\Gamma_1 \ \mathit{ctx} \qquad \Gamma_0.\text{🔒}.\Gamma_1 \vdash T \ \mathit{type}}{\Gamma_0.\text{🔒}.\Gamma_1.T \ \mathit{ctx}}$$
   In this case, our induction hypothesis tells us that both $\Gamma_0.\Gamma_1$ *ctx* and $\Gamma_0.\Gamma_1 \vdash T$ *type* hold. Therefore, we may apply the same rule to conclude that $\Gamma_0.\Gamma_1.T$ *ctx* holds as required.

   Case.
   $$\frac{\Gamma_0.\text{🔒}.\Gamma_1 \ \mathit{ctx}}{\Gamma_0.\text{🔒}.\Gamma_1.\text{🔒} \ \mathit{ctx}}$$
   In this case, our induction hypothesis tells us that $\Gamma_0.\Gamma_1$ *ctx* and we wish to show that $\Gamma_0.\Gamma_1.\text{🔒}$ *ctx*. However, this is immediate from our rules.

2. If $\Gamma_0.\text{🔒}.\Gamma_1 \vdash T$ *type* then $\Gamma_0.\Gamma_1 \vdash T$ *type*.

   Case.
   $$\frac{\Gamma_0.\text{🔒}.\Gamma_1 \vdash A \ \mathit{type} \qquad \Gamma_0.\text{🔒}.\Gamma_1.A \vdash B \ \mathit{type}}{\Gamma_0.\text{🔒}.\Gamma_1 \vdash \Pi(A, B) \ \mathit{type}}$$
   In this case, we have by induction hypothesis that $\Gamma_0.\Gamma_1 \vdash A$ *type* and $\Gamma_0.\Gamma_1.A \vdash B$ *type*. We wish to show that $\Gamma_0.\Gamma_1 \vdash \Pi(A, B)$ *type*. This, however, is again just rule.

3. If $\Gamma_0.\text{🔒}.\Gamma_1 \vdash T_0 = T_1$ *type* then $\Gamma_0.\Gamma_1 \vdash T_0 = T_1$ *type*.

   Case.
   $$\frac{\Gamma_0.\text{🔒}.\Gamma_1.\text{🔒} \vdash T_0 = T_1 \ \mathit{type}}{\Gamma_0.\text{🔒}.\Gamma_1 \vdash \Box T_0 = \Box T_1 \ \mathit{type}}$$
   We have, then, by induction hypothesis $\Gamma_0.\Gamma_1.\text{🔒} \vdash T_0 = T_1$ *type*. We wish to show that $\Gamma_0.\Gamma_1 \vdash \Box T_0 = \Box T_1$ *type*. This, again, immediately follows from our rule applied to our induction hypothesis.

4. If $\Gamma_0.\text{🔒}.\Gamma_1 \vdash t : T$ then $\Gamma_0.\Gamma_1 \vdash t : T$.

   Case.

$$\frac{\Gamma_0.\text{🔒}.\Gamma_1 \vdash T\ \mathit{type} \qquad \Gamma_0.\text{🔒}.\Gamma_1^{\text{🔓}} \vdash t : \Box T}{\Gamma_0.\text{🔒}.\Gamma_1 \vdash [t]_{\text{🔓}} : T}$$

   By induction hypothesis, we have $\Gamma_0.\Gamma_1^{\text{🔓}} \vdash t : \Box T$ and $\Gamma_0.\Gamma_1 \vdash T\ \mathit{type}$. We wish to show that $\Gamma_0.\Gamma_1 \vdash [t]_{\text{🔓}} : T$, but this is immediate from our rules.

5. If $\Gamma_0.\text{🔒}.\Gamma_1 \vdash t_0 = t_1 : T$ then $\Gamma_0.\Gamma_1 \vdash t_0 = t_1 : T$.

   Case.

$$\frac{\Gamma_0.\text{🔒}.\Gamma_1 \vdash t : \Box A}{\Gamma_0.\text{🔒}.\Gamma_1 \vdash [[t]_{\text{🔓}}]_{\text{🔒}} = t : \Box A}$$

   In this case, we have by induction hypothesis that $\Gamma_0.\Gamma_1 \vdash t : \Box A$. We wish to show that $\Gamma_0.\Gamma_1 \vdash [[t]_{\text{🔓}}]_{\text{🔒}} = t : \Box A$. We will do this by applying the same rule. However, our induction hypotheses are precisely the premises we need, so this is immediate.

6. If $\Gamma_0.\text{🔒}.\Gamma_1 \vdash \delta : \Delta$ then $\Gamma_0.\Gamma_1 \vdash \delta : \Delta$.

   Case.

$$\frac{\Gamma_0.\text{🔒}.\Gamma_1\ \mathit{ctx} \qquad \Gamma_2\ \mathit{ctx} \qquad \Gamma_0.\text{🔒}.\Gamma_1 \rhd_{\text{🔒}} \Gamma_2}{\Gamma_0.\text{🔒}.\Gamma_1 \vdash \mathsf{id} : \Gamma_2}$$

   In this case we have by induction hypothesis that $\Gamma_0.\Gamma_1\ \mathit{ctx}$ holds. Since $\Gamma_0.\text{🔒}.\Gamma_1 \rhd_{\text{🔒}} \Gamma_2$ holds we must then have $\Gamma_0.\Gamma_1 \rhd_{\text{🔒}} \Gamma_2$ and so we can apply same rule to conclude $\Gamma_0.\Gamma_1 \vdash \mathsf{id} : \Gamma_2$ as required.

   Case.

$$\frac{\Gamma_0.\text{🔒}.\Gamma_0'.\Gamma_1\ \mathit{ctx} \qquad \Delta\ \mathit{ctx} \qquad \Gamma_0.\text{🔒}.\Gamma_0' \rhd_{\text{🔒}} \Delta \qquad \text{🔒} \notin \Gamma_1 \qquad k = \|\Gamma_1\|}{\Gamma_0.\text{🔒}.\Gamma_1 \vdash \mathsf{p}^k : \Gamma_0.\text{🔒}.\Gamma_1}$$

   In this case we have by induction hypothesis that $\Gamma_0.\Gamma_0'.\Gamma_1\ \mathit{ctx}$ holds. Since $\Gamma_0.\text{🔒}.\Gamma_0' \rhd_{\text{🔒}} \Delta$ holds we must then have $\Gamma_0.\Gamma_0' \rhd_{\text{🔒}} \Delta$ and so we can apply same rule to conclude $\Gamma_0.\Gamma_0'.\Gamma_1 \vdash \mathsf{p}^k : \Delta$ as required.

   Case.

$$\frac{\Gamma_0.\text{🔒}.\Gamma_1\ \mathit{ctx} \qquad \Gamma_0.\text{🔒}.\Gamma_1^{\text{🔓}} \vdash \delta : \Delta}{\Gamma_0.\text{🔒}.\Gamma_1 \vdash \delta : \Delta.\text{🔒}}$$

   In this case we have by induction hypothesis that $\Gamma_0.\Gamma_1\ \mathit{ctx}$ holds. Since $\Gamma_0.\text{🔒}.\Gamma_1^{\text{🔓}} = \Gamma_0.\Gamma_1^{\text{🔓}}$ we then have $\Gamma_0.\Gamma_1^{\text{🔓}} \vdash \delta : \Delta$. We then obtain the desired conclusion by applying the same rule.

7. If $\Gamma_0.\text{🔒}.\Gamma_1 \vdash \delta_0 = \delta_1 : \Delta$ then $\Gamma_0.\Gamma_1 \vdash \delta_0 = \delta_1 : \Delta$.
   All cases follow immediately from our induction hypotheses. □

Lemma 1.2.5. *If* $\Gamma \vdash \mathcal{J}$ *then* $\Gamma^{\text{🔓}} \vdash \mathcal{J}$.

Proof. This follows by induction on the number of locks in Γ and by applying Theorem 1.2.4 at each step. □

Lemma 1.2.6.

1. *If* $\Gamma_0.\text{\faLock}.\Gamma_1$ *ctx then* $\Gamma_0.\text{\faLock}.\text{\faLock}.\Gamma_1$ *ctx.*
2. *If* $\Gamma_0.\text{\faLock}.\Gamma_1 \vdash T$ *type then* $\Gamma_0.\text{\faLock}.\text{\faLock}.\Gamma_1 \vdash T$ *type.*
3. *If* $\Gamma_0.\text{\faLock}.\Gamma_1 \vdash T_0 = T_1$ *type then* $\Gamma_0.\text{\faLock}.\text{\faLock}.\Gamma_1 \vdash T_0 = T_1$ *type.*
4. *If* $\Gamma_0.\text{\faLock}.\Gamma_1 \vdash t : T$ *then* $\Gamma_0.\text{\faLock}.\text{\faLock}.\Gamma_1 \vdash t : T$*.*
5. *If* $\Gamma_0.\text{\faLock}.\Gamma_1 \vdash t_0 = t_1 : T$ *then* $\Gamma_0.\text{\faLock}.\text{\faLock}.\Gamma_1 \vdash t_0 = t_1 : T$*.*
6. *If* $\Gamma_0.\text{\faLock}.\Gamma_1 \vdash \delta : \Delta$ *then* $\Gamma_0.\text{\faLock}.\text{\faLock}.\Gamma_1 \vdash \delta : \Delta$*.*
7. *If* $\Gamma_0.\text{\faLock}.\Gamma_1 \vdash \delta_0 = \delta_1 : \Delta$ *then* $\Gamma_0.\text{\faLock}.\text{\faLock}.\Gamma_1 \vdash \delta_0 = \delta_1 : \Delta$*.*

Proof. We proceed by mutual induction on the size of the input derivation. Every case of this follows immediately by the induction hypothesis. □

Theorem 1.2.7. *Supposing that* $\Gamma_0.\text{\faLock} \vdash A$ *type holds, the following facts are true.*

1. *If* $\Gamma_0.A.\text{\faLock}.\Gamma_1$ *ctx then* $\Gamma_0.\text{\faLock}.A.\Gamma_1$ *ctx.*
2. *If* $\Gamma_0.A.\text{\faLock}.\Gamma_1 \vdash T$ *type then* $\Gamma_0.\text{\faLock}.A.\Gamma_1 \vdash T$ *type.*
3. *If* $\Gamma_0.A.\text{\faLock}.\Gamma_1 \vdash T_0 = T_1$ *type then* $\Gamma_0.\text{\faLock}.A.\Gamma_1 \vdash T_0 = T_1$ *type.*
4. *If* $\Gamma_0.A.\text{\faLock}.\Gamma_1 \vdash t : T$ *then* $\Gamma_0.\text{\faLock}.A.\Gamma_1 \vdash t : T$*.*
5. *If* $\Gamma_0.A.\text{\faLock}.\Gamma_1 \vdash t_0 = t_1 : T$ *then* $\Gamma_0.\text{\faLock}.A.\Gamma_1 \vdash t_0 = t_1 : T$*.*
6. *If* $\Gamma_0.A.\text{\faLock}.\Gamma_1 \vdash \delta : \Delta$ *then* $\Gamma_0.\text{\faLock}.A.\Gamma_1 \vdash \delta : \Delta$*.*
7. *If* $\Gamma_0.A.\text{\faLock}.\Gamma_1 \vdash \delta_0 = \delta_1 : \Delta$ *then* $\Gamma_0.\text{\faLock}.A.\Gamma_1 \vdash \delta_0 = \delta_1 : \Delta$*.*

Proof. This proof mirrors the one of Theorem 1.2.4. It is done by simultaneous induction on all the judgments.

1. If $\Gamma_0.A.\text{\faLock}.\Gamma_1$ *ctx* then $\Gamma_0.\text{\faLock}.A.\Gamma_1$ *ctx*.
   For this branch, there is only one case that does not follow by induction: namely when $\Gamma_1 = \cdot$ and so we are considering $\Gamma_0.A.\text{\faLock}$ *ctx*. In this case, we have Γ_0 *ctx* and $\Gamma_0 \vdash A$ *type*. We wish to show that $\Gamma_0.\text{\faLock}.A$ *ctx*. First, we have $\Gamma_0.\text{\faLock}$ *ctx* immediately. In order to show that $\Gamma_0.\text{\faLock}.A$ *ctx* holds, however, we must show that $\Gamma_0.\text{\faLock} \vdash A$ *type* holds. This does not a-priori hold from what we have so far, however, we assumed it in the statement of this theorem and so we may conclude $\Gamma_0.\text{\faLock}.A$ *ctx*.

2. If $\Gamma_0.A.\text{\faLock}.\Gamma_1 \vdash T$ *type* then $\Gamma_0.\text{\faLock}.A.\Gamma_1 \vdash T$ *type*.
   Every single case of this part of the theorem is merely induction. To save time, therefore, I have presented only one case.

   Case.
   $$\frac{\Gamma_0.A.\text{\faLock}.\Gamma_1.\text{\faLock} \vdash T \textit{ type}}{\Gamma_0.A.\text{\faLock}.\Gamma_1 \vdash \Box T \textit{ type}}$$

   In this case, we have by induction hypothesis that $\Gamma_0.\text{\faLock}.A.\Gamma_1.\text{\faLock} \vdash T$ *type*. We wish to show $\Gamma_0.\text{\faLock}.A.\Gamma_1 \vdash \Box T$ *type*. This follows immediately by application of rule.

3. If $\Gamma_0.A.\text{\faLock}.\Gamma_1 \vdash T_0 = T_1$ *type* then $\Gamma_0.\text{\faLock}.A.\Gamma_1 \vdash T_0 = T_1$ *type*.
   This part of the theorem is identical to the case for $\Gamma_0.A.\text{\faLock}.\Gamma_1 \vdash T$ *type*.

4. If $\Gamma_0.A.\text{🔒}.\Gamma_1 \vdash t : T$ then $\Gamma_0.\text{🔒}.A.\Gamma_1 \vdash t : T$.

   Case.
$$\frac{\Gamma_0.A.\text{🔒}.\Gamma_1.\text{🔒} \vdash t : T}{\Gamma_0.A.\text{🔒}.\Gamma_1 \vdash [t]_{\text{🔒}} : \Box T}$$
   In this case, we have by induction hypothesis that $\Gamma_0.\text{🔒}.A.\Gamma_1.\text{🔒} \vdash t : T$. We wish to show that $\Gamma_0.\text{🔒}.A.\Gamma_1 \vdash [t]_{\text{🔒}} : \Box T$ holds. This follows immediately from the rule for $[-]_{\text{🔒}}$.

   Case.
$$\frac{\Gamma.A.\text{🔒}.\Gamma_1 \vdash T \ \mathit{type} \qquad (\Gamma_0.A.\text{🔒}.\Gamma_1)^{\text{🔓}} \vdash t : \Box T}{\Gamma_0.A.\text{🔒}.\Gamma_1 \vdash [t]_{\text{🔓}} : T}$$
   In this case, we have by induction hypothesis that $(\Gamma_0.\text{🔒}.A.\Gamma_1)^{\text{🔓}} \vdash t : \Box T$ and $\Gamma_0.\text{🔒}.A.\Gamma_1 \vdash T \ \mathit{type}$. We wish to show that $\Gamma_0.\text{🔒}.A.\Gamma_1 \vdash [t]_{\text{🔓}} : T$ holds. This follows immediately from the rule for $[-]_{\text{🔓}}$.

5. If $\Gamma_0.A.\text{🔒}.\Gamma_1 \vdash t_0 = t_1 : T$ then $\Gamma_0.\text{🔒}.A.\Gamma_1 \vdash t_0 = t_1 : T$.

   Case.
$$\frac{\Gamma_0.A.\text{🔒}.\Gamma_1 \vdash t : \Box A}{\Gamma_0.A.\text{🔒}.\Gamma_1 \vdash [[t]_{\text{🔓}}]_{\text{🔒}} = t : \Box A}$$
   In this case we have by induction hypothesis that $\Gamma_0.\text{🔒}.A.\Gamma_1 \vdash t : \Box A$. Therefore, by application of our rules we have $\Gamma_0.\text{🔒}.A.\Gamma_1 \vdash [[t]_{\text{🔓}}]_{\text{🔒}} = t : \Box A$

   Case.
$$\frac{(\Gamma_0.A.\text{🔒}.\Gamma_1)^{\text{🔓}}.\text{🔒} \vdash t : A}{\Gamma_0.A.\text{🔒}.\Gamma_1 \vdash [[t]_{\text{🔒}}]_{\text{🔓}} = t : A}$$
   We need to show $\Gamma_0.\text{🔒}.A.\Gamma_1 \vdash [[t]_{\text{🔒}}]_{\text{🔓}} = t : A$; applying the same rule, it suffices to show that $(\Gamma_0.\text{🔒}.A.\Gamma_1)^{\text{🔓}}.\text{🔒} \vdash t : A$. Observing that $(\Gamma_0.\text{🔒}.A.\Gamma_1)^{\text{🔓}} = (\Gamma_0.A.\text{🔒}.\Gamma_1)^{\text{🔓}}$, we see that we can just use our existing premise.

6. If $\Gamma_0.A.\text{🔒}.\Gamma_1 \vdash \delta : \Delta$ then $\Gamma_0.\text{🔒}.A.\Gamma_1 \vdash \delta : \Delta$.

   Case.
$$\frac{\Gamma_0.A.\text{🔒}.\Gamma_1 \ \mathit{ctx} \qquad \Delta \ \mathit{ctx} \qquad \Gamma_0.A.\text{🔒}.\Gamma_1 \rhd_{\text{🔒}} \Delta}{\Gamma_0 \vdash \mathsf{id} : \Delta}$$
   In this case we have $\Gamma_0.\text{🔒}.A.\Gamma_1 \ \mathit{ctx}$ and $\Delta \ \mathit{ctx}$. It therefore suffices to show that $\Gamma_0.\text{🔒}.A.\Gamma_1 \rhd_{\text{🔒}} \Delta$. However, this follows from the fact that $\Gamma_0.A.\text{🔒}.\Gamma_1 \rhd_{\text{🔒}} \Delta$ holds. Therefore, we are done by applying the rule for id.

7. If $\Gamma_0.A.\text{🔒}.\Gamma_1 \vdash \delta_0 = \delta_1 : \Delta$ then $\Gamma_0.\text{🔒}.A.\Gamma_1 \vdash \delta_0 = \delta_1 : \Delta$.
   All cases here follow from the induction hypotheses. □

Lemma 1.2.8. *If Γ ctx and $\Gamma^{\text{🔓}}.\text{🔒} \vdash \mathcal{J}$ then $\Gamma.\text{🔒} \vdash \mathcal{J}$.*

Proof. This follows by induction on Γ and by applying Theorems 1.2.4 and 1.2.7 and Lemma 1.2.6 at each step. □

In order to prove the remaining facts, we first need the following “lifting theorem” regarding substitutions.

Lemma 1.2.9. *If* $\Gamma \vdash \delta : \Delta$ *then* $\Gamma^{\text{🔓}} \vdash \delta : \Delta^{\text{🔓}}$

Proof. We proceed by induction on the derivation of $\Gamma \vdash \delta : \Delta$.

Case.

$$\frac{\Gamma_0\ \mathit{ctx} \qquad \Gamma_1\ \mathit{ctx} \qquad \Gamma_0 \triangleright_{\text{🔒}} \Gamma_1}{\Gamma_0 \vdash \mathsf{id} : \Gamma_1}$$

It is simple to see by induction that if $\Gamma_0 \triangleright_{\text{🔒}} \Gamma_1$ holds then $\Gamma_0^{\text{🔓}} = \Gamma_1^{\text{🔓}}$. Since, by Lemma 1.2.5, we have $\Gamma_0^{\text{🔓}}\ \mathit{ctx}$ we then have $\Gamma_0^{\text{🔓}} \vdash \mathsf{id} : \Gamma_1^{\text{🔓}}$ immediately by applying this rule.

Case.

$$\frac{\Gamma\ \mathit{ctx} \qquad \Delta\ \mathit{ctx} \qquad \cdot \triangleright_{\text{🔒}} \Delta}{\Gamma \vdash \cdot : \Delta}$$

In this case, we have no induction hypothesis and our goal is to show that $\Gamma^{\text{🔓}} \vdash \cdot : \Delta^{\text{🔓}}$. Simple induction tells us that $\Delta^{\text{🔓}} = \cdot$. Therefore, we merely need to show $\Gamma^{\text{🔓}} \vdash \cdot : \cdot$ and this follows from immediately from our rule together with Lemma 1.2.5.

Case.

$$\frac{\Delta \vdash T\ \mathit{type} \qquad \Gamma \vdash \delta : \Delta \qquad \Gamma \vdash t : T[\delta]}{\Gamma \vdash \delta.t : \Delta.T}$$

In this case, our induction hypothesis states that $\Gamma^{\text{🔓}} \vdash \delta : \Delta^{\text{🔓}}$ and we wish to show that $\Gamma^{\text{🔓}} \vdash \delta.t : \Delta.T^{\text{🔓}}$. First, we note that $\Delta.T^{\text{🔓}} = \Delta^{\text{🔓}}.T$. Thus, we apply the rule for adjoining a term to a substitution. We must show that the following hold:

- $\Gamma^{\text{🔓}} \vdash t : T[\delta]$
- $\Gamma^{\text{🔓}} \vdash t : \Delta^{\text{🔓}}$
- $(\Delta.T)^{\text{🔓}}\ \mathit{ctx}$ (which is equivalent to $\Delta^{\text{🔓}}.T\ \mathit{ctx}$)

However, we have the first by assumption and Lemma 1.2.5, the next is our induction hypothesis and the last follows again from Lemma 1.2.5 and our assumption that $\Delta.T\ \mathit{ctx}$.

Case.

$$\frac{\Gamma_0 \vdash \delta_0 : \Gamma_1 \qquad \Gamma_1 \vdash \delta_1 : \Gamma_2}{\Gamma_0 \vdash \delta_1 \circ \delta_0 : \Gamma_2}$$

By induction hypothesis we have $\Gamma_0^{\text{🔓}} \vdash \delta_0 : \Gamma_1^{\text{🔓}}$ and $\Gamma_1^{\text{🔓}} \vdash \delta_1 : \Gamma_2^{\text{🔓}}$. However, we then just apply the composition rule again to obtain $\Gamma_0^{\text{🔓}} \vdash \delta_1 \circ \delta_0 : \Gamma_2^{\text{🔓}}$ as required.

Case.

$$\frac{\Gamma_0\ \mathit{ctx} \qquad \Gamma_0^{\text{🔓}} \vdash \delta : \Gamma_1}{\Gamma_0 \vdash \delta : \Gamma_1.\text{🔒}}$$

By induction hypothesis, we have that $\Gamma_0^{\text{🔓}} \vdash \delta : \Gamma_1^{\text{🔓}}$. However, since $\Gamma_1.\text{🔒}^{\text{🔓}} = \Gamma_1^{\text{🔓}}$ this immediately gives us the desired conclusion when Lemma 1.2.5 is applied to $\Gamma_0\ \mathit{ctx}$.

Case.

$$\frac{\Gamma_0.\Gamma_1\ \mathit{ctx} \qquad \Delta\ \mathit{ctx} \qquad k = \|\Gamma_1\| \qquad \Gamma_0 \triangleright_{\text{🔒}} \Delta \qquad \text{🔒} \notin \Gamma_1}{\Gamma_0.\Gamma_1 \vdash \mathsf{p}^k : \Delta}$$

In this case, we have no induction hypothesis but we will show that $\Gamma_0.\Gamma_1^{\text{🔓}} \vdash \mathsf{p}^k : \Delta^{\text{🔓}}$ by application of the same rule. We have that $\text{🔒} \notin \Gamma_1^{\text{🔓}}$ and $\|\Gamma_1\| = k$ immediately. All we need to show is that $\Gamma_0.\Gamma_1^{\text{🔓}}\ \mathit{ctx}$ and $\Gamma_0^{\text{🔓}} \triangleright_{\text{🔒}} \Delta^{\text{🔓}}$. The first follows from Lemma 1.2.5 and our assumption that $\Gamma_0.\Gamma_1\ \mathit{ctx}$. The second follows from the fact that we must have $\Gamma_0^{\text{🔓}} = \Delta^{\text{🔓}}$ as $\Gamma_0 \triangleright_{\text{🔒}} \Delta$ holds. $\square$

Lemma 1.2.10. *If* $\Gamma \vdash \delta_0 = \delta_1 : \Delta$ *then* $\Gamma^{\text{🔓}} \vdash \delta_0 = \delta_1 : \Delta^{\text{🔓}}$

Proof. Proceeds by induction on the derivation and follows directly from Lemmas 1.2.5 and 1.2.9. □

Lemma 1.2.11. *If* $\Gamma \vdash \delta : \Delta.\text{🔒}$ *then* $\Gamma^{\text{🔓}} \vdash \delta : \Delta$.

Proof. We proceed by induction on $\Gamma \vdash \delta : \Delta.\text{🔒}$. Only a few cases apply:

Case.

$$\frac{\Delta\ \mathit{ctx} \qquad \Gamma.\text{🔒}\ \mathit{ctx} \qquad \Delta \rhd_{\text{🔒}} \Gamma.\text{🔒}}{\Delta \vdash \mathsf{id} : \Gamma.\text{🔒}}$$

In this case we wish to show $\Delta^{\text{🔓}} \vdash \mathsf{id} : \Gamma$ but this is immediate by Lemma 1.2.5.

Case.

$$\frac{\Delta\ \mathit{ctx} \qquad \Gamma.\text{🔒}\ \mathit{ctx} \qquad \cdot \rhd_{\text{🔒}} \Gamma.\text{🔒}}{\Delta \vdash \cdot : \Gamma.\text{🔒}}$$

In this case we wish to show $\Delta^{\text{🔓}} \vdash \cdot : \Gamma$. However, it must be that $\cdot \rhd_{\text{🔒}} \Gamma$ by simple induction. Therefore, we have our goal by applying the same rule and using Lemma 1.2.5.

Case.

$$\frac{\Gamma_0 \vdash \delta_0 : \Gamma_1 \qquad \Gamma_1 \vdash \delta_1 : \Gamma_2.\text{🔒}}{\Gamma_0 \vdash \delta_1 \circ \delta_0 : \Gamma_2.\text{🔒}}$$

In this case we wish to show $\Gamma_0^{\text{🔓}} \vdash \delta_1 \circ \delta_0 : \Gamma_2$. We have $\Gamma_1^{\text{🔓}} \vdash \delta_1 : \Gamma_2$ by induction hypothesis. By Lemma 1.2.8 and $\Gamma_0 \vdash \delta_0 : \Gamma_1$ we have $\Gamma_0^{\text{🔓}} \vdash \delta_0 : \Gamma_1^{\text{🔓}}$. Therefore, by the rule for composition we have $\Gamma_0^{\text{🔓}} \vdash \delta_1 \circ \delta_0 : \Gamma_2$ as required.

Case.

$$\frac{\Gamma_0\ \mathit{ctx} \qquad \Gamma_0^{\text{🔓}} \vdash \delta : \Gamma_1}{\Gamma_0 \vdash \delta : \Gamma_1.\text{🔒}}$$

In this case we wish to show $\Gamma_0^{\text{🔓}} \vdash \delta : \Gamma_1$ but this is immediate by assumption.

Case.

$$\frac{\Gamma_0.\Gamma_1\ \mathit{ctx} \qquad k = \|\Gamma_1\| \qquad \Gamma_0 \rhd_{\text{🔒}} \Gamma_0' \qquad \text{🔒} \notin \Gamma_1}{\Gamma_0.\Gamma_1 \vdash \mathsf{p}^k : \Gamma_0'.\text{🔒}}$$

In this case we wish to show to show $\Gamma_0.\Gamma_1^{\text{🔓}} \vdash \mathsf{p}^k : \Gamma_0'.\text{🔒}$. However, we have that $\Gamma_0.\Gamma_1^{\text{🔓}}$ *ctx* by Lemma 1.2.5 and $\Gamma_0^{\text{🔓}} \rhd_{\text{🔒}} \Gamma_0'$ by definition. Finally, $\|\Gamma_1^{\text{🔓}}\| = \|\Gamma_1\|$ so the goal is immediate. □

Lemma 1.2.12. *Suppose* $\Delta \vdash \delta : \Gamma_0.\text{🔒}.T.\Gamma_1$ *and* $\Gamma_0.T.\text{🔒}.\Gamma_1$ *ctx, then* $\Delta \vdash \delta : \Gamma_0.T.\text{🔒}.\Gamma_1$

Proof. We proceed by induction over the input derivation.

Subcase.

$$\frac{\Gamma\ \mathit{ctx} \qquad \Delta_0.\text{🔒}.T.\Delta_1\ \mathit{ctx} \qquad \cdot \rhd_{\text{🔒}} \Delta_0.\text{🔒}.T.\Delta_1}{\Gamma \vdash \cdot : \Delta_0.\text{🔒}.T.\Delta_1}$$

In this case we have a contradiction: $\cdot \rhd_{\text{🔒}} \Delta_0.\text{🔒}.T.\Delta_1$ cannot hold.

Subcase.

$$\frac{\Gamma_0\ \mathit{ctx} \qquad \Delta_0.\text{🔒}.T.\Delta_1\ \mathit{ctx} \qquad \Gamma_0 \rhd_{\text{🔒}} \Delta_0.\text{🔒}.T.\Delta_1}{\Gamma_0 \vdash \mathsf{id} : \Delta_0.\text{🔒}.T.\Delta_1}$$

We wish to show $\Gamma_0 \vdash \mathsf{id} : \Delta_0.T.\text{🔒}.\Delta_1$. We have $\Delta_0.T.\text{🔒}.\Delta_1$ *ctx* by assumption. Furthermore we have $\Delta_0.\text{🔒}.T.\Delta_1 \rhd_{\text{🔒}} \Delta_0.T.\text{🔒}.\Delta_1$. Therefore our goal follows immediately from the same rule and the fact that $- \rhd_{\text{🔒}} -$ is transitive.

Subcase.

$$\frac{\Delta \vdash T\ \mathit{type} \qquad \Gamma \vdash \delta : \Delta \qquad \Gamma \vdash t : T[\delta]}{\Gamma \vdash \delta.t : \Delta.T}$$

Now there are two cases to consider here, either $\Delta = \Delta'.\faLock$ and we wish to prove $\Gamma \vdash \delta.t : \Delta'.T.\faLock$ or $\Delta = \Delta_0.\faLock.T'.\Delta_1$ and we wish to prove $\Gamma \vdash \delta.t : \Delta_0.T'.\faLock.\Delta_1.T$.

Recall that we also have $\Delta'.T.\faLock\ \mathit{ctx}$ in the first case and $\Delta_0.T'.\faLock.\Delta_1.T\ \mathit{ctx}$ in the second case.

In the first case, we observe that it suffices to show $\Gamma^{\faUnlock} \vdash \delta.t : \Delta'.T$. For this, we observe that we have by assumption that $\Delta'.\faLock \vdash T\ \mathit{type}$ and so $\Delta' \vdash T\ \mathit{type}$ must hold by Theorem 1.2.4. We have that $\Gamma^{\faUnlock} \vdash t : T[\delta]$ from our assumption and Lemma 1.2.5. Finally, we must show $\Gamma^{\faUnlock} \vdash \delta : \Delta'$ but this follows from Lemma 1.2.11.

For the second case, we have by induction hypothesis $\Gamma \vdash \delta : \Delta_0.T'.\faLock.\Delta_1$. We also have that $\Delta_0.T'.\faLock.\Delta_1 \vdash T\ \mathit{type}$ from $\Delta_0.T'.\faLock.\Delta_1.T\ \mathit{ctx}$. Therefore, we may apply the same rule to obtain the desired goal.

Subcase.

$$\frac{\Gamma_0 \vdash \delta_0 : \Gamma_1 \qquad \Gamma_1 \vdash \delta_1 : \Gamma_2}{\Gamma_0 \vdash \delta_1 \circ \delta_0 : \Gamma_2}$$

This is immediate by induction hypothesis.

Subcase.

$$\frac{\Gamma_0\ \mathit{ctx} \qquad \Gamma_0^{\faUnlock} \vdash \delta : \Gamma_1}{\Gamma_0 \vdash \delta : \Gamma_1.\faLock}$$

This is immediate by induction hypothesis.

Subcase.

$$\frac{\Gamma_0.\Gamma_1\ \mathit{ctx} \qquad \Delta_0.\faLock.T.\Delta_1\ \mathit{ctx} \qquad \Gamma_0 \rhd_{\faLock} \Delta_0.\faLock.T.\Delta_1 \qquad k = \|\Gamma_1\| \qquad \faLock \notin \Gamma_1}{\Gamma_0.\Gamma_1 \vdash \mathsf{p}^k : \Delta_0.\faLock.T.\Delta_1}$$

We wish to show $\Gamma_0.\Gamma_1 \vdash \mathsf{p}^k : \Delta_0.T.\faLock.\Delta_1$. We have by assumption that $\Gamma_0.\Gamma_1\ \mathit{ctx}$ and $\Delta_0.T.\faLock.\Delta_1\ \mathit{ctx}$ hold. Furthermore, we know that $\Delta_0.\faLock.T.\Delta_1 \rhd_{\faLock} \Delta_0.T.\faLock.\Delta_1$ holds by definition. The goal then follows immediately from the same rule and the fact that $- \rhd_{\faLock} -$ is transitive. □

Lemma 1.2.13. *Suppose $\Delta \vdash \delta : \Gamma_0.\faLock.\faLock.\Gamma_1$ and $\Gamma_0.\faLock.\Gamma_1$ ctx then $\Delta \vdash \delta : \Gamma_0.\faLock.\Gamma_1$*

Proof. This is immediate by induction on the input derivation from the fact that the $-^{\faUnlock}$ is idempotent. □

Lemma 1.2.14. *Suppose $\Delta \vdash \delta : \Gamma_0.\Gamma_1$ and $\Gamma_0.\faLock.\Gamma_1$ ctx, then $\Delta \vdash \delta : \Gamma_0.\faLock.\Gamma_1$*

Proof. This is immediate by induction on the input derivation and from Lemma 1.2.5. □

Lemma 1.2.15. *If $\Gamma_1 \vdash \mathsf{id} : \Gamma_2$ then the following facts hold.*

1. *If Γ_0 ctx and $\Gamma_0 \vdash \delta : \Gamma_1$ then $\Gamma_0 \vdash \delta : \Gamma_2$.*
2. *For any Γ if $\Gamma_i.\Gamma$ ctx and $\Gamma_2.\Gamma \vdash \mathcal{J}$ then $\Gamma_1.\Gamma \vdash \mathcal{J}$.*

Proof. This proof proceeds by induction on the derivation of $\Gamma_1 \vdash \mathsf{id} : \Gamma_2$.

Case.

$$\frac{\Gamma_1\ \mathit{ctx} \qquad \Gamma_2\ \mathit{ctx} \qquad \Gamma_1 \rhd_{\faLock} \Gamma_2}{\Gamma_1 \vdash \mathsf{id} : \Gamma_2}$$

1. This is just an application of Lemmas 1.2.12 to 1.2.14.
2. This is just an application of Theorems 1.2.4 and 1.2.7 and Lemma 1.2.6.

Case.

$$\frac{\Gamma_1 \ \mathit{ctx} \qquad \Gamma_1^{\text{🔓}} \vdash \mathsf{id} : \Gamma_2'}{\Gamma_1 \vdash \mathsf{id} : \Gamma_2'.\text{🔒}}$$

In this case we have by induction hypothesis that the following facts hold:

- If Γ_0 *ctx* and $\Gamma_0 \vdash \delta : \Gamma_1^{\text{🔓}}$ then $\Gamma_0 \vdash \delta : \Gamma_2'$.
- For any Γ if $\Gamma_2'.\Gamma \vdash \mathcal{J}$, $\Gamma_1.\Gamma$ *ctx*, and $\Gamma_2'.\Gamma$ *ctx*, then $\Gamma_1^{\text{🔓}}.\Gamma \vdash \mathcal{J}$.

We wish to show the following:

- If Γ_0 *ctx* and $\Gamma_0 \vdash \delta : \Gamma_1$ then $\Gamma_0 \vdash \delta : \Gamma_2'.\text{🔒}$.
- For any Γ if $\Gamma_2'.\text{🔒}.\Gamma \vdash \mathcal{J}$ $\Gamma_1.\Gamma$ *ctx*, and $\Gamma_2'.\text{🔒}.\Gamma$ *ctx*, then $\Gamma_1.\Gamma \vdash \mathcal{J}$.

For the first item, we observe that if $\Gamma_0 \vdash \delta : \Gamma_1$ then $\Gamma_0^{\text{🔓}} \vdash \delta : \Gamma_1'^{\text{🔓}}$ from Lemma 1.2.9. Next, we then have by our induction hypothesis that $\Gamma_0^{\text{🔓}} \vdash \delta : \Gamma_2'$ since $\Gamma_0^{\text{🔓}}$ *ctx* by Lemma 1.2.5. Next, from straightforward application of our rules we have $\Gamma_0 \vdash \delta : \Gamma_2'.\text{🔒}$ as required.

For the second item, suppose that $\Gamma_2'.\text{🔒}.\Gamma \vdash \mathcal{J}$ for some Γ. We wish to show that $\Gamma_1.\Gamma \vdash \mathcal{J}$. In order to show this, we instantiate our induction hypothesis with $\text{🔒}.\Gamma$. We then have $\Gamma_1^{\text{🔓}}.\text{🔒}.\Gamma \vdash \mathcal{J}$. By Lemma 1.2.8 and Theorem 1.2.4 we then have $\Gamma_1.\Gamma \vdash \mathcal{J}$. □

Theorem 1.2.16.

1. *If $\Gamma \vdash T$ type then Γ ctx.*
2. *If $\Gamma \vdash t : T$ then $\Gamma \vdash T$ type.*
3. *If $\Gamma_1 \vdash \delta : \Gamma_2$ then Γ_i ctx.*
4. *If $\Gamma \vdash T_1 = T_2$ type then $\Gamma \vdash T_i$ type.*
5. *If $\Gamma \vdash t_1 = t_2 : T$ then $\Gamma \vdash t_i : T$.*
6. *If $\Gamma \vdash \delta_1 = \delta_2 : \Delta$ then $\Gamma \vdash \delta_i : \Delta$.*

Proof. This theorem is largely standard *except* for the cases concerning substitutions and $\Box$. We therefore only show these cases.

1. If $\Gamma \vdash T$ *type* then Γ *ctx*.

   Case.

   $$\frac{\Gamma.\text{🔒} \vdash A \ \mathit{type}}{\Gamma \vdash \Box A \ \mathit{type}}$$

   In this case we have by induction hypothesis that $\Gamma.\text{🔒}$ *ctx*. We wish to show that Γ *ctx* however this follows by induction on the derivation of $\Gamma.\text{🔒}$ *ctx*.

2. If $\Gamma \vdash t : T$ then $\Gamma \vdash T$ *type*.

Case.

$$\frac{\Gamma \vdash A\ \mathit{type} \qquad \Gamma^{\text{🔓}} \vdash t : \Box A}{\Gamma \vdash [t]_{\text{🔓}} : A}$$

In this case we have by assumption that $\Gamma \vdash A\ \mathit{type}$. Notice that this assumption is necessary here because we only have by induction hypothesis that $\Gamma^{\text{🔓}} \vdash \Box A\ \mathit{type}$. Since this could have come from the universe rule, it is difficult to obtain $\Gamma^{\text{🔓}}.\text{🔒} \vdash A\ \mathit{type}$ which would give us the conclusion.

Case.

$$\frac{\Gamma.\text{🔒} \vdash t : A}{\Gamma \vdash [t]_{\text{🔒}} : \Box A}$$

In this case we have by induction hypothesis that $\Gamma.\text{🔒} \vdash A\ \mathit{type}$. Therefore, by rule we have the goal: $\Gamma \vdash \Box A\ \mathit{type}$.

3. If $\Gamma_1 \vdash \delta : \Gamma_2$ then $\Gamma_i\ \mathit{ctx}$.

Case.

$$\frac{\Gamma\ \mathit{ctx} \qquad \Delta\ \mathit{ctx} \qquad \cdot \rhd_{\text{🔒}} \Delta}{\Gamma \vdash \cdot : \Delta}$$

In this case we have $\Gamma\ \mathit{ctx}$ and $\Delta\ \mathit{ctx}$ and we wish to show that $\Gamma\ \mathit{ctx}$ and $\Delta\ \mathit{ctx}$. Immediate.

Case.

$$\frac{\Delta\ \mathit{ctxT} \qquad \Gamma \vdash \delta : \Delta \qquad \Gamma \vdash t : T[\delta]}{\Gamma \vdash \delta.t : \Delta.T}$$

In this case we have $\Gamma\ \mathit{ctx}$ by induction hypothesis and $\Delta.T\ \mathit{ctx}$ by assumption. We wish to show that $\Gamma\ \mathit{ctx}$ and $\Delta.T\ \mathit{ctx}$. Immediate.

Case.

$$\frac{\Gamma_1 \vdash \delta_1 : \Gamma_2 \qquad \Gamma_2 \vdash \delta_2 : \Gamma_3}{\Gamma_1 \vdash \delta_2 \circ \delta_1 : \Gamma_3}$$

In this case we have $\Gamma_1\ \mathit{ctx}$ by induction hypothesis and $\Gamma_3\ \mathit{ctx}$ by assumption. We wish to show that $\Gamma_1\ \mathit{ctx}$ and $\Gamma_3\ \mathit{ctx}$. Immediate.

Case.

$$\frac{\Gamma_1\ \mathit{ctx} \qquad \Gamma_1^{\text{🔓}} \vdash \delta : \Gamma_2}{\Gamma_1 \vdash \delta : \Gamma_2.\text{🔒}}$$

In this case we have $\Gamma_2\ \mathit{ctx}$ by induction hypothesis and $\Gamma_1\ \mathit{ctx}$ by assumption. This is precisely the goal however.

Case.

$$\frac{\Gamma_1.\Gamma_2\ \mathit{ctx} \qquad \Delta\ \mathit{ctx} \qquad \Gamma_1 \rhd_{\text{🔒}} \Delta \qquad k = \|\Gamma_2\| \qquad \text{🔒} \notin \Gamma_2}{\Gamma_1.\Gamma_2 \vdash \mathrm{p}^k : \Delta}$$

In this case we have $\Gamma_1.\Gamma_2\ \mathit{ctx}$ and $\Delta\ \mathit{ctx}$ by assumption.

4. If $\Gamma \vdash T_1 = T_2\ \mathit{type}$ then $\Gamma \vdash T_i\ \mathit{type}$.

Case.

$$\frac{\Gamma \vdash \delta : \Delta \qquad \Delta.\text{🔒} \vdash A\ \mathit{type}}{\Gamma \vdash (\Box A)[\delta] = \Box(A[\delta])\ \mathit{type}}$$

In this case we must show that $\Gamma \vdash (\Box A)[\delta]$ *type* and $\Gamma \vdash \Box(A[\delta])$ *type*. The first one follows immediately by application of rules since $\Delta \vdash \Box A$ *type* follows directly from our assumptions. For the second, first observe that $\Gamma.\text{🔒} \vdash \delta : \Delta.\text{🔒}$ by application of rule and Lemma 1.2.5. Therefore, $\Gamma.\text{🔒} \vdash A[\delta]$ *type* and so $\Gamma \vdash \Box(A[\delta])$ *type*

5. If $\Gamma \vdash t_1 = t_2 : T$ then $\Gamma \vdash t_i : T$.

Case.

$$\frac{\Gamma^{\text{🔓}}.\text{🔒} \vdash t : A}{\Gamma \vdash [[t]_{\text{🔒}}]_{\text{🔓}} = t : A}$$

In this case, we wish to show that $\Gamma \vdash t : A$ and $\Gamma \vdash [[t]_{\text{🔒}}]_{\text{🔓}} : A$. In order to do this, first observe that by Lemma 1.2.8 we have $\Gamma.\text{🔒} \vdash t : A$. Therefore, by Theorem 1.2.4 there is a proof that $\Gamma \vdash t : A$. For the second goal, we apply the intro rule for $[-]_{\text{🔓}}$ so we must show $\Gamma^{\text{🔓}} \vdash [t]_{\text{🔒}} : \Box A$. However, this follows from $\Gamma^{\text{🔓}}.\text{🔒} \vdash t : A$ which is precisely our assumption.

Case.

$$\frac{\Gamma.\text{🔒} \vdash A \; type \qquad \Gamma \vdash t : \Box A}{\Gamma \vdash [[t]_{\text{🔓}}]_{\text{🔒}} = t : \Box A}$$

In this case we wish to show that $\Gamma \vdash t : \Box A$ and $\Gamma \vdash [[t]_{\text{🔓}}]_{\text{🔒}} : \Box A$. The first is immediate by assumption. For the second, we must show that $\Gamma \vdash [[t]_{\text{🔓}}]_{\text{🔒}} : \Box A$. By application of the introduction rules, it suffices to show that $\Gamma^{\text{🔓}} \vdash t : A$. However, this follows from Lemma 1.2.5 applied to $\Gamma \vdash t : A$.

Case.

$$\frac{\Gamma \vdash \delta : \Delta \qquad \Delta.\text{🔒} \vdash t : T}{\Gamma \vdash [t]_{\text{🔒}}[\delta] = [t[\delta]]_{\text{🔒}} : (\Box T)[\delta]}$$

In this case, we wish to show that $\Gamma \vdash [t]_{\text{🔒}}[\delta] : (\Box T)[\delta]$ and $\Gamma \vdash [t[\delta]]_{\text{🔒}} : (\Box T)[\delta]$.
For the first one, we see by the application of the $[-]_{\text{🔒}}$ rule that $\Delta \vdash [t]_{\text{🔒}} : \Box T$. Next, we have by the explicit substitution rule that $\Gamma \vdash [t]_{\text{🔒}}[\delta] : (\Box T)[\delta]$.
For the second goal, we note that we have by Lemma 1.2.5 that $\Gamma^{\text{🔓}} \vdash \delta : \Delta$. Therefore, we have $\Gamma.\text{🔒} \vdash \delta : \Delta.\text{🔒}$ immediately. We can then apply the explicit substitution rule to conclude that $\Gamma.\text{🔒} \vdash t : T[\delta]$. Next, we apply the rule for $[-]_{\text{🔒}}$ to get $\Gamma \vdash [t]_{\text{🔒}} : \Box(T[\delta])$. Finally, we observe that by the conversion rule we then have $\Gamma \vdash [t]_{\text{🔒}} : (\Box T)[\delta]$.

Case.

$$\frac{\Gamma \vdash \delta : \Delta \qquad \Delta^{\text{🔓}} \vdash t : \Box T}{\Gamma \vdash [t]_{\text{🔓}}[\delta] = [t[\delta]]_{\text{🔓}} : T[\delta]}$$

In this case, we wish to show that $\Gamma \vdash [t]_{\text{🔓}}[\delta] : T[\delta]$ and $\Gamma \vdash [t[\delta]]_{\text{🔓}} : T[\delta]$.
For the first one, we see by the application of the $[-]_{\text{🔓}}$ rule that $\Delta \vdash [t]_{\text{🔓}} : T$. Next, we have by the explicit substitution rule that $\Gamma \vdash [t]_{\text{🔓}}[\delta] : T[\delta]$.
For the second goal, we note that we have by Lemma 1.2.9 that $\Gamma^{\text{🔓}} \vdash \delta : \Delta^{\text{🔓}}$. We can then apply the explicit substitution rule to conclude the following: $\Gamma^{\text{🔓}} \vdash t : T[\delta]$. Next, we apply the rule for $[-]_{\text{🔓}}$ to get $\Gamma \vdash [t]_{\text{🔓}} : T[\delta]$.

6. If $\Gamma \vdash \delta_1 = \delta_2 : \Delta$ then $\Gamma \vdash \delta_i : \Delta$.

Case.

$$\frac{\Gamma_1 \vdash \delta_1 : \Gamma_2 \qquad \Gamma_2 \vdash \delta_2 : \Gamma_3 \qquad \Gamma_3 \vdash \delta_3 : \Gamma_4}{\Gamma_1 \vdash \delta_3 \circ (\delta_2 \circ \delta_1) = (\delta_3 \circ \delta_2) \circ \delta_1 : \Gamma_4}$$

In this case we must show that $\Gamma_1 \vdash \delta_3 \circ (\delta_2 \circ \delta_1) : \Gamma_4$ and $\Gamma_1 \vdash (\delta_3 \circ \delta_2) \circ \delta_1 : \Gamma_4$. We have by assumption that $\Gamma_i \vdash \delta_i : \Gamma_{i+1}$, so both of these cases are immediate by the rule for composition.

Case.

$$\frac{\Gamma_1 \vdash \delta : \Gamma_2 \qquad \Gamma_2 \vdash \mathsf{id} : \Gamma_3}{\Gamma_1 \vdash \mathsf{id} \circ \delta = \delta : \Gamma_3}$$

In this case, we wish to show $\Gamma_1 \vdash \mathsf{id} \circ \delta : \Gamma_3$ and $\Gamma_1 \vdash \delta : \Gamma_3$. We have by assumption that $\Gamma_1 \vdash \delta : \Gamma_2$ and $\Gamma_2 \vdash \mathsf{id} : \Gamma_3$. The first goal is immediate by the rule for composition. For the second goal, we use Lemma 1.2.15 to conclude that $\Gamma_1 \vdash \delta : \Gamma_3$.

Case.

$$\frac{\Gamma_1 \vdash \mathsf{id} : \Gamma_2 \qquad \Gamma_2 \vdash \delta : \Gamma_3}{\Gamma_1 \vdash \delta \circ \mathsf{id} = \delta : \Gamma_3}$$

In this case, we wish to show $\Gamma_1 \vdash \delta \circ \mathsf{id} : \Gamma_3$ and $\Gamma_1 \vdash \delta : \Gamma_3$. We have by assumption that $\Gamma_2 \vdash \delta : \Gamma_3$ and $\Gamma_1 \vdash \mathsf{id} : \Gamma_2$. The first goal is immediate by the rule for composition. For the second goal, we use Lemma 1.2.15 to conclude that $\Gamma_1 \vdash \delta : \Gamma_3$.

Case.

$$\frac{\Gamma_1 \vdash \delta_1 : \Gamma_2 \qquad \Gamma_2 \vdash \delta_2.t : \Gamma_3}{\Gamma_1 \vdash (\delta_2.t) \circ \delta_1 = (\delta_2 \circ \delta_1).(t[\delta_1]) : \Gamma_3}$$

We have by assumption that $\Gamma_1 \vdash \delta_1 : \Gamma_2$ and $\Gamma_2 \vdash \delta_2.t : \Gamma_3$. We wish to show $\Gamma_1 \vdash (\delta_2.t) \circ \delta_1 : \Gamma_3$ and $\Gamma_1 \vdash (\delta_2 \circ \delta_1).(t[\delta_1]) : \Gamma_3$. The first goal is immediate from our assumptions and the rule for composition. We focus then on the second goal.

In order to show this, we proceed by induction on $\Gamma_2 \vdash \delta_2.t : \Gamma_3$.

Case.

$$\frac{\Gamma_3'.T\ \mathit{ctx} \qquad \Gamma_2 \vdash \delta_2 : \Gamma_3' \qquad \Gamma_2 \vdash t : T[\delta_2]}{\Gamma_2 \vdash \delta_2.t : \Gamma_3'.T}$$

In this case, we wish to show the following:

$$\Gamma_1 \vdash (\delta_2 \circ \delta_1).(t[\delta_1]) : \Gamma_3'.T$$

First, observe that by the rule for composition we have $\Gamma_1 \vdash \delta_2 \circ \delta_1 : \Gamma_3'$. Next, by the rule for explicit substitutions, we have $\Gamma_2 \vdash t[\delta_1] : T[\delta_2][\delta_1]$ and so by conversion, $\Gamma_2 \vdash t[\delta_1] : T[\delta_2 \circ \delta_1]$. Therefore, by the rule for extension: $\Gamma_1 \vdash (\delta_2 \circ \delta_1).(t[\delta_1]) : \Gamma_3'.T$ as required.

Case.

$$\frac{\Gamma_2\ \mathit{ctx} \qquad \Gamma_2^{\text{🔓}} \vdash \delta_2.t : \Gamma_3'}{\Gamma_2 \vdash \delta_2.t : \Gamma_3'.\text{🔒}}$$

In this case, we have by induction hypothesis that the following holds:

$$\Gamma_2^{\text{🔓}} \vdash (\delta_2 \circ \delta_1).t[\delta_1] : \Gamma_3'$$

Therefore, we have $\Gamma_2 \vdash (\delta_2 \circ \delta_1).t[\delta_1] : \Gamma_3'.\text{🔒}$ from application of our rules.

Case.

$$\frac{\Gamma_1 \vdash \mathsf{p}^{n+1} : \Gamma_2}{\Gamma_1 \vdash \mathsf{p}^{n+1} = \mathsf{p}^{n} \circ \mathsf{p}^{1} : \Gamma_2}$$

In this case we have by assumption that $\Gamma_1 \vdash \mathsf{p}^{n+1} : \Gamma_2$ and we wish to show $\Gamma_1 \vdash \mathsf{p}^{n+1} : \Gamma_2$ and $\Gamma_1 \vdash \mathsf{p}^{n} \circ \mathsf{p}^{1} : \Gamma_2$. The first of these conclusions is immediate. For the second goal, we proceed by induction on $\Gamma_1 \vdash \mathsf{p}^{n+1} : \Gamma_2$.

Case.

$$\frac{\Gamma_1'.\Gamma_1''\ ctx \qquad \Delta\ ctx \qquad \Gamma_1' \rhd_{\text{🔒}} \Delta \qquad n+1 = \|\Gamma_1''\| \qquad \text{🔒} \notin \Gamma_1''}{\Gamma_1'.\Gamma_1'' \vdash \mathsf{p}^{n+1} : \Delta}$$

Note that here $\Delta = \Gamma_2$.

In this case, note that $\Gamma_1'' = \Xi.T$ for some Ξ of length n. We can therefore derive that $\Gamma_1'.\Gamma_1'' \vdash \mathsf{p}^1 : \Gamma_1'.\Xi$ and $\Gamma_1'.\Xi \vdash \mathsf{p}^n : \Delta$. By the rules for composition we have the desired goal.

Case.

$$\frac{\Gamma_1\ ctx \qquad \Gamma_1^{\text{🔓}} \vdash \mathsf{p}^{n+1} : \Gamma_2'}{\Gamma_1 \vdash \mathsf{p}^{n+1} : \Gamma_2'.\text{🔒}}$$

In this case, we have by induction hypothesis that $\Gamma_1^{\text{🔓}} \vdash \mathsf{p}^n \circ \mathsf{p}^1 : \Gamma_2'.\text{🔒}$. We then have $\Gamma_1 \vdash \mathsf{p}^n \circ \mathsf{p}^1 : \Gamma_2'.\text{🔒}$ by applying a rule.

Case.

$$\frac{\Gamma_1 \vdash \delta.t : \Gamma_2 \qquad \Gamma_2 \vdash \mathsf{p}^1 : \Gamma_3}{\Gamma_1 \vdash \mathsf{p}^1 \circ (\delta.t) = \delta : \Gamma_3}$$

In this case, we have by $\Gamma_1 \vdash \delta.t : \Gamma_2$ and $\Gamma_2 \vdash \mathsf{p}^1 : \Gamma_3$. We wish to show $\Gamma_1 \vdash \mathsf{p}^1 \circ (\delta.t) : \Gamma_3$ and $\Gamma_1 \vdash \delta : \Gamma_3$. The first goal is immediate from our assumptions. We merely need to show the latter.

In order to show this, we will show by induction on the size of the derivation $\Gamma_2 \vdash \mathsf{p}^1 : \Gamma_3$ that if $\Gamma_1 \vdash \delta.t : \Gamma_2$ then $\Gamma_1 \vdash \delta : \Gamma_3$.

We proceed by case on the derivation of $\Gamma_1 \vdash \delta.t : \Gamma_2$.

Subcase.

$$\frac{\Gamma_2'.T\ ctx \qquad \Gamma_1 \vdash \delta : \Gamma_2' \qquad \Gamma_1 \vdash t : T[\delta]}{\Gamma_1 \vdash \delta.t : \Gamma_2'.T}$$

In this case, we have $\Gamma_1 \vdash \delta : \Gamma_2'$. We now need to show that $\Gamma_1 \vdash \delta : \Gamma_3$. In order to do this, we will prove that $\Gamma_2' \vdash \mathsf{id} : \Gamma_3$ by induction on $\Gamma_2'.T \vdash \mathsf{p}^1 : \Gamma_3$. The result will then Lemma 1.2.15.

Subsubcase.

$$\frac{\Gamma_2'.T\ ctx \qquad \Delta\ ctx \qquad \Gamma_2' \rhd_{\text{🔒}} \Delta}{\Gamma_2'.T \vdash \mathsf{p}^1 : \Delta}$$

In this case, we observe that we are trying to show $\Gamma_2' \vdash \mathsf{id} : \Delta$ but this is immediate from the assumptions we have and the rule for id.

Subsubcase.

$$\frac{\Gamma_2\ ctx \qquad \Gamma_2'^{\text{🔓}}.T \vdash \mathsf{p}^1 : \Gamma_3'}{\Gamma_2'.T \vdash \mathsf{p}^1 : \Gamma_3'.\text{🔒}}$$

In this case, we have $\Gamma_2^{\text{🔓}} \vdash \mathsf{id} : \Gamma_3'$ and so we have $\Gamma_2 \vdash \mathsf{id} : \Gamma_3'.\text{🔒}$ from our assumption of $\Gamma_2\ ctx$ and the same rule.

Subcase.

$$\frac{\Gamma_1\ ctx \qquad \Gamma_1^{\text{🔓}} \vdash \delta.t : \Gamma_2'}{\Gamma_1 \vdash \delta.t : \Gamma_2'.\text{🔒}}$$

In this case we have $\Gamma_2'.\text{🔒} \vdash \mathsf{p}^1 : \Gamma_3$ and we wish to show $\Gamma_1 \vdash \delta : \Gamma_3$. Inversion on the former tells us that it must be that $\Gamma_3 = \Gamma_3'.\text{🔒}$ and that there is a strictly smaller derivation $\Gamma_2^{\text{🔓}} \vdash \mathsf{p}^1 : \Gamma_3'$.

Therefore, it suffices to show $\Gamma_1^{\text{🔓}} \vdash \delta : \Gamma_3'$ in order to establish our goal. We know that $\Gamma_1^{\text{🔓}} \vdash \delta.t : \Gamma_2'^{\text{🔓}}$ by Lemma 1.2.9. We then apply our induction hypothesis we our strictly smaller derivation of $\Gamma_2^{\text{🔓}} \vdash \mathsf{p}^1 : \Gamma_3'$. □

2 Computing in $\mathsf{MLTT}_{🔒}$

2.1 Semantic domain

We now define the semantic domains in which $\mathsf{MLTT}_{🔒}$ programs compute. We diverge from the standard presentation of normalization by evaluation in terms of partial applicative structures by actively distinguishing between closure instantiation and the partial application operation. Colors are used to distinguish between all the different domains; the color of an identifier is part of its lexical meaning, making A, A distinct metavariables.

$$\begin{array}{llcl}
\textit{(values)} & A, u & ::= & \uparrow^{A} e \mid \lambda(f) \mid \mathbf{\Pi}(A, B) \mid \mathbf{zero} \mid \mathbf{succ}(v) \mid \mathbf{nat} \mid \langle v_1, v_2 \rangle \mid \mathbf{\Sigma}(A, B) \\
 & & & \Box A \mid \mathbf{shut}(v) \mid \mathbf{U}_i \mid \mathbf{Id}(A, v_1, v_2) \mid \mathbf{refl}(v) \\
\textit{(neutrals)} & e & ::= & \mathrm{var}_k \mid e.\mathbf{app}(d) \mid e.\mathbf{fst} \mid e.\mathbf{snd} \mid e.\mathbf{open} \mid e.\mathbf{natrec}(A, v, f) \\
 & & & e.\mathbf{J}(C, f, A, v_1, v_2) \\
\textit{(environments)} & \rho & ::= & \cdot \mid \rho.v \\
\textit{(closures)} & A, f & ::= & t \triangleleft \rho \\
\textit{(normals)} & d & ::= & \downarrow^{A} v
\end{array}$$

2.2 Semantic partial operations

Elements of the semantic domains are animated through partial operations, such as evaluation of terms, application of values, etc. In this section, we define the graphs of these partial operations inductively.

$$\boxed{[\![t]\!]_\rho = v}$$

$$\textsc{eval/var}\ \frac{\rho(i) = v}{[\![\mathsf{var}_i]\!]_\rho = v} \qquad \textsc{eval/nat}\ \frac{}{[\![\mathsf{nat}]\!]_\rho = \mathbf{nat}} \qquad \textsc{eval/zero}\ \frac{}{[\![\mathsf{zero}]\!]_\rho = \mathbf{zero}} \qquad \textsc{eval/succ}\ \frac{[\![t]\!]_\rho = u}{[\![\mathsf{succ}(t)]\!]_\rho = \mathbf{succ}(u)}$$

$$\textsc{eval/natrec}\ \frac{[\![z]\!]_\rho = v_z \qquad [\![n]\!]_\rho = n \qquad \underline{\mathbf{natrec}}(A \triangleleft \rho, v_z, s \triangleleft \rho, n) = v}{[\![\mathsf{natrec}(A, z, s, n)]\!]_\rho = v} \qquad \textsc{eval/pi}\ \frac{[\![A]\!]_\rho = A}{[\![\Pi(A, B)]\!]_\rho = \mathbf{\Pi}(A, B \triangleleft \rho)}$$

$$\textsc{eval/lam}\ [\![\lambda(t)]\!]_\rho = \lambda(t \triangleleft \rho) \qquad \textsc{eval/app}\ \frac{[\![s]\!]_\rho = u \qquad [\![t]\!]_\rho = v \qquad \underline{\mathbf{app}}(u, v) = w}{[\![s(t)]\!]_\rho = w} \qquad \textsc{eval/sig}\ \frac{[\![A]\!]_\rho = A}{[\![\Sigma(A, B)]\!]_\rho = \mathbf{\Sigma}(A, B \triangleleft \rho)}$$

$$\textsc{eval/fst}\ \frac{[\![t]\!]_\rho = v \qquad \underline{\mathbf{fst}}(v) = v_1}{[\![\mathsf{fst}(t)]\!]_\rho = v_1} \qquad \textsc{eval/snd}\ \frac{[\![t]\!]_\rho = v \qquad \underline{\mathbf{snd}}(v) = v_2}{[\![\mathsf{snd}(t)]\!]_\rho = v_2} \qquad \textsc{eval/uni}\ \frac{}{[\![\mathsf{U}_i]\!]_\rho = \mathbf{U}_i} \qquad \textsc{eval/box}\ \frac{[\![A]\!]_\rho = A}{[\![\Box A]\!]_\rho = \Box A}$$

$$\frac{[\![t]\!]_\rho = v}{[\![[t]_{🔒}]\!]_\rho = \mathbf{shut}(v)}\ \textsc{eval/shut} \qquad \frac{[\![t]\!]_\rho = v \qquad \underline{\mathbf{open}}(v) = v'}{[\![[t]_{🔓}]\!]_\rho = v'}\ \textsc{eval/open} \qquad \frac{[\![\delta]\!]_\rho = \rho' \qquad [\![t]\!]_{\rho'} = v}{[\![t[\delta]]\!]_\rho = v}\ \textsc{eval/esubst}$$

$$\frac{[\![T]\!]_\rho = A \qquad [\![t_0]\!]_\rho = v_0 \qquad [\![t_i]\!]_\rho = v_i}{[\![\mathsf{Id}(T, t_0, t_1)]\!]_\rho = \mathbf{Id}(A, v_0, v_1)} \qquad \frac{[\![t]\!]_\rho = v}{[\![\mathsf{refl}(t)]\!]_\rho = \mathbf{refl}(v)}$$

$$\frac{[\![t_2]\!]_\rho = u \qquad \underline{\mathbf{J}}(C \lhd \rho, t_1 \lhd \rho, u) = v}{[\![\mathsf{J}(C, t_1, t_2)]\!]_\rho = v}$$

$$\boxed{f[w_1, \dots, w_n] = v}$$

$$\frac{[\![t]\!]_{\rho.w_1 \dots w_n} = v}{(t \lhd \rho)[w_1, \dots, w_n] = v}\ \textsc{inst/clo}$$

$$\boxed{[\![\delta]\!]_{\rho_1} = \rho_2}$$

$$\frac{}{[\![\mathsf{id}]\!]_\rho = \rho}\ \textsc{eval/id} \qquad \frac{}{[\![\cdot]\!]_\rho = \cdot}\ \textsc{eval/emp} \qquad \frac{[\![\delta]\!]_{\rho_1} = \rho_2 \qquad [\![t]\!]_{\rho_1} = v}{[\![\delta.t]\!]_{\rho_1} = \rho_2.v}\ \textsc{eval/ext} \qquad \frac{}{[\![\mathsf{p}^n]\!]_{\rho.v_1 \dots v_n} = \rho}\ \textsc{eval/proj}$$

$$\frac{[\![\delta_1]\!]_{\rho_1} = \rho_2 \qquad [\![\delta_2]\!]_{\rho_2} = \rho_3}{[\![\delta_2 \circ \delta_1]\!]_{\rho_1} = \rho_3}\ \textsc{eval/compose}$$

$$\boxed{\underline{\mathbf{app}}(u, v) = w}$$

$$\frac{f[v] = w}{\underline{\mathbf{app}}(\lambda(f), v) = w}\ \textsc{app/lam} \qquad \frac{B[v] = B_v}{\underline{\mathbf{app}}(\uparrow^{\Pi(A,B)} e, v) = \uparrow^{B_v} e.\mathbf{app}(\downarrow^A v)}\ \textsc{app/shift}$$

$$\boxed{\underline{\mathbf{J}}(C, f, v) = u}$$

$$\frac{f[v] = u}{\underline{\mathbf{J}}(C, f, \mathbf{refl}(v)) = u}\ \textsc{J/refl} \qquad \frac{C[u_1, u_2, \uparrow^{\mathbf{Id}(A,u_1,u_2)} e] = B}{\underline{\mathbf{J}}(C, f, \uparrow^{\mathbf{Id}(A,u_1,u_2)} e) = \uparrow^B e.\mathbf{J}(C, f, A, u_1, u_2)}\ \textsc{J/shift}$$

$$\boxed{\underline{\mathbf{fst}}(v) = v_1}$$

$$\frac{}{\underline{\mathbf{fst}}(\langle v_1, v_2 \rangle) = v_1} \qquad \frac{}{\underline{\mathbf{fst}}(\uparrow^{\Sigma(A,B)} e) = \uparrow^A e.\mathbf{fst}}$$

$$\boxed{\underline{\mathbf{snd}}(v) = v_2}$$

$$\frac{}{\underline{\mathbf{snd}}(\langle v_1, v_2 \rangle) = v_2} \qquad \frac{B[\uparrow^A e.\mathbf{fst}] = B'}{\underline{\mathbf{snd}}(\uparrow^{\Sigma(A,B)} e) = \uparrow^{B'} e.\mathbf{snd}}$$

$$\boxed{\underline{\mathbf{natrec}}(A, v_z, f_s, n) = v}$$

$$\frac{}{\underline{\mathbf{natrec}}(A, v_z, f_s, \mathbf{zero}) = v_z} \qquad \frac{\underline{\mathbf{natrec}}(A, v_z, f_s, n) = v_p \qquad f_s[n, v_p] = v}{\underline{\mathbf{natrec}}(A, v_z, f_s, \mathbf{succ}(n)) = v}$$

$$\frac{A[n] = A'}{\underline{\mathbf{natrec}}(A, v_z, f_s, \uparrow^{\mathbf{nat}} e) = \uparrow^{A'} e.\mathbf{natrec}(A, v_z, f_s)}$$

$$\boxed{\underline{\mathbf{open}}(v_1) = v_2}$$

$$\frac{}{\underline{\mathbf{open}}(\mathbf{shut}(v)) = v} \qquad \frac{}{\underline{\mathbf{open}}(\uparrow^{\square A} e) = \uparrow^{A} e.\mathbf{open}}$$

$$\boxed{\lceil d \rceil_n = t}$$

RB/FUN

$$\frac{B[\mathbf{var}_n] = B' \qquad \underline{\mathbf{app}}(v, \uparrow^{A} \mathbf{var}_n) = b \qquad \lceil \downarrow^{B'} b \rceil_{n+1} = t}{\lceil \downarrow^{\Pi(A,B)} v \rceil_n = \lambda(t)}$$

RB/PAIR

$$\frac{\underline{\mathbf{fst}}(v) = l \qquad \underline{\mathbf{snd}}(v) = r \qquad B[l] = B' \qquad \lceil \downarrow^{A} l \rceil_n = t_1 \qquad \lceil \downarrow^{B'} r \rceil_n = t_2}{\lceil \downarrow^{\Sigma(A,B)} v \rceil_n = (t_1, t_2)}$$

RB/REFL

$$\frac{\lceil \downarrow^{A} u \rceil_n = t}{\lceil \downarrow^{\mathbf{Id}(A, v_1, v_2)} \mathbf{refl}(u) \rceil_n = \mathsf{refl}(t)}$$

RB/ZERO

$$\frac{}{\lceil \downarrow^{\mathbf{nat}} \mathbf{zero} \rceil_n = \mathsf{zero}}$$

RB/SUCC

$$\frac{\lceil \downarrow^{\mathbf{nat}} v \rceil_n = t}{\lceil \downarrow^{\mathbf{nat}} \mathbf{succ}(v) \rceil_n = \mathsf{succ}(t)}$$

RB/SHUT

$$\frac{\underline{\mathbf{open}}(v) = v' \qquad \lceil \downarrow^{A} v' \rceil_n = t}{\lceil \downarrow^{\square A} v \rceil_n = [t]_{🔒}}$$

RB/NAT/NE

$$\frac{\lceil e \rceil_n = t}{\lceil \downarrow^{\mathbf{nat}} \uparrow^{B} e \rceil_n = t}$$

RB/ID/NE

$$\frac{\lceil e \rceil_n = t}{\lceil \downarrow^{\mathbf{Id}(A, u_0, u_1)} \uparrow^{B} e \rceil_n = t}$$

RB/NE

$$\frac{\lceil e \rceil_n = t}{\lceil \downarrow^{\uparrow^{C} e'} \uparrow^{B} e \rceil_n = t}$$

RB/TP

$$\frac{\lceil v \rceil_n^{\mathsf{ty}} = A}{\lceil \downarrow^{\mathbf{U}_i} v \rceil_n = A}$$

$$\boxed{\lceil e \rceil_n = t}$$

RB/APP

$$\frac{\lceil e \rceil_n = s \qquad \lceil d \rceil_n = t}{\lceil e.\mathbf{app}(d) \rceil_n = s(t)}$$

RB/VAR

$$\frac{}{\lceil \mathbf{var}_k \rceil_n = \mathsf{var}_{n-(k+1)}}$$

RB/FST

$$\frac{\lceil e \rceil_n = t}{\lceil e.\mathbf{fst} \rceil_n = \mathsf{fst}(t)}$$

RB/SND

$$\frac{\lceil e \rceil_n = t}{\lceil e.\mathbf{snd} \rceil_n = \mathsf{snd}(t)}$$

RB/J

$$\frac{\begin{array}{l} C[\mathbf{var}_n, \mathbf{var}_{n+1}, \mathbf{var}_{n+2}] = C_g \\ \lceil C_g \rceil^{\mathsf{ty}}_{n+3} = C \qquad C[\mathbf{var}_n, \mathbf{var}_n, \mathbf{refl}(n)] = C_r \qquad f[\mathbf{var}_n] = v \qquad \lceil \downarrow^{C_r} v \rceil_{n+1} = t_1 \qquad \lceil e \rceil_n = t_2 \end{array}}{\lceil e.\mathbf{J}(C, f, A, u_1, u_2) \rceil_n = \mathsf{J}(C, t_1, t_2)}$$

$$\frac{\lceil e \rceil_n = t}{\lceil e.\mathbf{open} \rceil_n = [t]_{🔓}}\ \text{RB/OPEN}$$

$$\frac{A[\uparrow^{\mathrm{nat}} \mathbf{var}_n] = A' \qquad \lceil A' \rceil^{\mathsf{ty}}_{n+1} = A \qquad A[\mathbf{zero}] = A_z \qquad \lceil \downarrow^{A_z} v_z \rceil_n = z \qquad A[\mathbf{succ}(\mathbf{var}_n)] = A_s \qquad f_s[\uparrow^{\mathrm{nat}} \mathbf{var}_n, \uparrow^{A'} \mathbf{var}_{n+1}] = v_s \qquad \lceil \downarrow^{A_s} v_s \rceil_{n+2} = s \qquad \lceil e \rceil_n = m}{\lceil e.\mathbf{natrec}(A, v_z, f_s) \rceil_n = \mathsf{natrec}(A, z, s, m)}\ \text{RB/NATREC}$$

$$\boxed{\lceil v \rceil^{\mathsf{ty}}_n = t}$$

$$\frac{\lceil e \rceil_n = t}{\lceil \uparrow^A e \rceil^{\mathsf{ty}}_n = t}\ \text{RB/VAL/NE} \qquad \frac{}{\lceil \mathbf{nat} \rceil^{\mathsf{ty}}_n = \mathsf{nat}}\ \text{RB/NAT} \qquad \frac{\lceil A \rceil^{\mathsf{ty}}_n = A \qquad B[\uparrow^A \mathbf{var}_n] = B' \qquad \lceil B' \rceil^{\mathsf{ty}}_{n+1} = B}{\lceil \mathbf{\Pi}(A, B) \rceil^{\mathsf{ty}}_n = \Pi(A, B)}\ \text{RB/PI}$$

$$\frac{\lceil A \rceil^{\mathsf{ty}}_n = T \qquad \lceil \downarrow^A v_1 \rceil_n = t_1 \qquad \lceil \downarrow^A v_2 \rceil_n = t_2}{\lceil \mathbf{Id}(A, v_1, v_2) \rceil^{\mathsf{ty}}_n = \mathsf{Id}(T, t_1, t_2)}\ \text{RB/ID} \qquad \frac{\lceil A \rceil^{\mathsf{ty}}_n = A \qquad B[\uparrow^A \mathbf{var}_n] = B' \qquad \lceil B' \rceil^{\mathsf{ty}}_{n+1} = B}{\lceil \mathbf{\Sigma}(A, B) \rceil^{\mathsf{ty}}_n = \Sigma(A, B)}\ \text{RB/SIG}$$

$$\frac{\lceil A \rceil^{\mathsf{ty}}_n = A}{\lceil \square A \rceil^{\mathsf{ty}}_n = \square A}\ \text{RB/BOX} \qquad \frac{}{\lceil \mathbf{U}_i \rceil_n = \mathsf{U}_i}\ \text{RB/UNI}$$

Reflecting contexts

Context length $\|\Gamma\|$ is the number of cells in the context, not including locks. A context is reflected as follows:

$$\boxed{\uparrow\Gamma = \rho}$$

$$\text{REFLECT/EMP}\ \ \uparrow\cdot = \cdot \qquad \frac{\uparrow\Gamma = \rho \qquad [\![T]\!]_\rho = A}{\uparrow\Gamma.T = \rho.\uparrow^A \mathbf{var}_{\|\Gamma\|}}\ \text{REFLECT/SNOC/VAR} \qquad \frac{\uparrow\Gamma = \rho}{\uparrow\Gamma.🔒 = \rho}\ \text{REFLECT/SNOC/LOCK}$$

The full normalization algorithm

The full algorithm is then defined as follows:

$$\frac{\uparrow\Gamma = \rho \qquad [\![A]\!]_\rho = A \qquad [\![t]\!]_\rho = v \qquad \lceil \downarrow^A v \rceil_{\|\Gamma\|} = t'}{\underline{\mathbf{nbe}}^A_\Gamma(t) = t'}$$

Miscellaneous lemmas

Lemma 2.2.1. *Suppose* $[\![M]\!]_\rho = v$, *and* ρ' *is an extension of the environment* ρ *such that* $|\rho'| - |\rho| = m$. *Then also* $[\![M[\mathrm{p}^m]]\!]_{\rho'} = v$.

Proof. $[\![M[\mathrm{p}^m]]\!]_{\rho'} = v$ holds if $[\![\mathrm{p}^m]\!]_{\rho'} = \rho''$ and $[\![M]\!]_{\rho''} = v$. Observe that $[\![\mathrm{p}^m]\!]_{\rho'} = \rho$ because $\rho' = \rho.v_1...v_m$. Next, we have by assumption $[\![M]\!]_\rho = v$ we therefore may conclude $[\![M[\mathrm{p}^m]]\!]_{\rho'} = v$ as required. ∎

2.3 Determinism

At this point it is possible to prove determinism of the judgments by simple induction. In all situations there should only be one applicable rule. This does not guarantee termination or that the algorithm is in any way correct, but it justifies the abuse of notation we shall adopt from now on. Henceforth we will write partial functions for several of the judgments. For instance, we fix the following notations:

- $\underline{\mathbf{open}}(u)$ for the unique v such that $\underline{\mathbf{open}}(u) = v$ when such a v exists;
- $f[v]$ for the unique u such that $f[v] = u$;
- $\underline{\mathbf{fst}}(v)$ for the unique u such that $\underline{\mathbf{fst}}(v) = u$;
- $\underline{\mathbf{snd}}(v)$ for the unique u such that $\underline{\mathbf{snd}}(v) = u$;
- $\underline{\mathbf{app}}(v_0, v_1)$ for the unique u such that $\underline{\mathbf{app}}(v_1, v_2) = u$;

We will also write $[\![t]\!]_\rho$ for the unique result, v, of $[\![t]\!]_\rho = v$ and likewise $[\![\delta]\!]_\rho = \rho'$ when $[\![\delta]\!]_\rho = \rho'$.

3 Completeness of Normalization

The correctness of the normalization algorithm defined in Chapter 2 is split into two main parts: completeness and soundness. Completeness is proved by constructing a model of $\mathsf{MLTT}_{🔒}$ in *partial equivalence relations* (PERs), and soundness is proved using a logical relations argument that glues the PER model together with the syntax of $\mathsf{MLTT}_{🔒}$.

3.1 PER model

Neutrals and normals

The main lemma used to establish completeness is that every type specifies a PER which lies between the PERs of neutrals and normals, which we define below.

$$\frac{\forall n.\ \exists t.\ \lceil e_0 \rceil_n = t \land \lceil e_1 \rceil_n = t}{e_0 \sim e_1 \in \mathcal{Ne}} \qquad \frac{\forall n.\ \exists t.\ \lceil d_0 \rceil_n = t \land \lceil d_1 \rceil_n = t}{d_0 \sim d_1 \in \mathcal{Nf}} \qquad \frac{\forall n.\ \exists A.\ \lceil A_0 \rceil_n^{\mathsf{ty}} = A \land \lceil A_1 \rceil_n^{\mathsf{ty}} = A}{A_0 \sim A_1 \in \mathcal{Ty}}$$

PERs for types

We construct a model of type theory in Kripke partial equivalence relations over an arbitrary non-empty poset $\mathbb{P}$; the main part of the construction is to develop a countable hierarchy of type universes, which we do in a style which first appeared in in Allen [All87], and has been used in three successful formalization efforts [AR14; WB18; SH18a].

The construction of the type hierarchy can be seen as an instance of induction-recursion[1], but we find it more clear to work concretely in terms of fixed-points on the complete lattice of subsets of the product of values (types) and binary relations on values in our domain indexed over $\mathbb{P}$. The indexing allows us to model $\square$ in an interesting and nontrivial way. We begin by defining a few of the critical domains for our construction:

$$\begin{aligned}
\mathbf{Rel} &= \mathcal{P}(\mathbb{P} \times \mathbf{Val} \times \mathbf{Val}) && \text{(step-indexed relation)}\\
\mathbf{SFam} &= \mathbb{P} \longrightarrow \mathbf{Rel} && \text{(indexed relations)}\\
\mathbf{Fam} &= \mathbf{Val} \times \mathbf{Val} \longrightarrow \mathbf{Rel} && \text{(family of relations)}\\
\mathbf{Sys} &= \mathcal{P}(\mathbb{P} \times \mathbf{Val} \times \mathbf{Val} \times \mathbf{Rel}) && \text{(type system)}
\end{aligned}$$

Next, we define some notation for working with these domains:

$$\frac{\tau(n, A_0, A_1, R)}{\tau \models_n A_0 \sim A_1 \downarrow R} \qquad \frac{\exists R.\ \tau \models_n A_0 \sim A_1 \downarrow R}{\tau \models_n A_0 \sim A_1} \qquad \frac{R(n, v_0, v_1)}{n \Vdash v_0 \sim v_1 \in R}$$

$$\frac{\forall m \leq n.\ \forall v_0, v_1.\ m \Vdash v_0 \sim v_1 \in R \implies \tau \models_m B_0[v_0] \sim B_1[v_1] \downarrow S}{\tau \models_n R \gg B_0 \sim B_1 \downarrow S}$$

[1] In fact, it was the first instance!

Notation 3.1.1 (Fiber of a relation)**.** For an indexed relation $R \in \mathbf{Rel}$, we will often write R_n for its fiber $\{(u_0, u_1) \mid n \Vdash u_0 \sim u_1 \in R\}$.

Definition 3.1.2 (Partial equivalence relation)**.** An indexed relation $R \in \mathbf{Rel}$ is called symmetric when each fiber R_n is a symmetric relation on $\mathbf{Val} \times \mathbf{Val}$; likewise, it is called transitive when each fiber is transitive. R is called a partial equivalence relation (PER) when it is both symmetric and transitive.

Definition 3.1.3 (Monotonicity)**.** A relation $R \in \mathbf{Rel}$ is called *monotone* iff whenever $m \leq n$, then $R_n \subseteq R_m$.

Definition 3.1.4 (Compatibility)**.** A relation $R \in \mathbf{Rel}$ is *compatible for* (A_0, A_1) if the following two properties hold:

1. If $e_0 \sim e_1 \in \mathcal{Ne}$ then $n \Vdash \uparrow^{A_0} e_0 \sim \uparrow^{A_1} e_1 \in R$ for all n.
2. If $n \Vdash v_0 \sim v_1 \in R$ then $\downarrow^{A_0} v_0 \sim \downarrow^{A_1} v_1 \in \mathcal{Nf}$.

We shall say a relation $R \in \mathbf{Rel}$ is *compatible for types* if the following two conditions hold:

1. If $e_0 \sim e_1 \in \mathcal{Ne}$ then $n \Vdash \uparrow^{\mathsf{U}_i} e_0 \sim \uparrow^{\mathsf{U}_i} e_1 \in R$ for all n and i.
2. If $n \Vdash v_0 \sim v_1 \in R$ then $v_0 \sim v_1 \in \mathcal{Ty}$.

Constructions on relations

We begin by separately developing some constructions on indexed binary relations; we define these for arbitrary indexed relations and families of relations, rather than requiring beforehand that we have a monotone PER.

$$
\begin{aligned}
[\![\Pi]\!] &\in \mathbf{Rel} \to \mathbf{Fam} \to \mathbf{Rel} \\
[\![\Sigma]\!] &\in \mathbf{Rel} \to \mathbf{Fam} \to \mathbf{Rel} \\
[\![\square]\!] &\in \mathbf{Rel} \to \mathbf{Rel} \\
[\![\mathsf{Id}]\!] &\in \mathbf{Rel} \to \mathbf{Val} \to \mathbf{Val} \to \mathbf{Rel} \\
[\![\mathbb{N}]\!] &\in \mathbf{Rel}
\end{aligned}
$$

These are defined as the least relations closed under the following rules:

$$\frac{S : \mathbf{Fam} \qquad \forall m \leq n.\ \forall v_0, v_1.\ m \Vdash v_0 \sim v_1 \in R \implies m \Vdash \underline{\mathbf{app}}(u_0, v_0) \sim \underline{\mathbf{app}}(u_1, v_1) \in S(v_0, v_1)}{n \Vdash u_0 \sim u_1 \in [\![\Pi]\!](R, S)}$$

$$\frac{S : \mathbf{Fam} \qquad n \Vdash \underline{\mathbf{fst}}(u_0) \sim \underline{\mathbf{fst}}(u_1) \in R \qquad n \Vdash \underline{\mathbf{snd}}(u_0) \sim \underline{\mathbf{snd}}(u_1) \in S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_1))}{n \Vdash u_0 \sim u_1 \in [\![\Sigma]\!](R, S)}$$

$$\frac{\forall m.\ m \Vdash \underline{\mathbf{open}}(u_0) \sim \underline{\mathbf{open}}(u_1) \in R}{n \Vdash u_0 \sim u_1 \in [\![\square]\!](R)} \qquad \frac{m \Vdash u_0 \sim v_0 \in R \qquad m \Vdash v_0 \sim v_1 \in R \qquad m \Vdash v_1 \sim u_1 \in R}{n \Vdash \mathbf{refl}(v_0) \sim \mathbf{refl}(v_1) \in [\![\mathsf{Id}]\!](R, u_0, u_1)}$$

$$\frac{e_0 \sim e_1 \in \mathcal{Ne}}{n \Vdash \uparrow^{\mathbf{Id}(A_0, v_0, v_1)} e_0 \sim \uparrow^{\mathbf{Id}(A_1, w_0, w_1)} e_1 \in [\![\mathsf{Id}]\!](R, u_0, u_1)} \qquad \frac{}{n \Vdash \mathbf{zero} \sim \mathbf{zero} \in [\![\mathbb{N}]\!]}$$

$$\frac{n \Vdash u_0 \sim u_1 \in [\![\mathbb{N}]\!]}{n \Vdash \mathbf{succ}(u_0) \sim \mathbf{succ}(u_1) \in [\![\mathbb{N}]\!]} \qquad \frac{e_0 \sim e_1 \in \mathcal{Ne}}{n \Vdash \uparrow^{\mathbf{nat}} e_0 \sim \uparrow^{\mathbf{nat}} e_1 \in [\![\mathbb{N}]\!]}$$

Lemma 3.1.5. *For any $R \in \mathbf{Rel}$ and $S \in \mathbf{Fam}$, the relation $[\![\Pi]\!](R, S)$ is monotone.*

Proof. Suppose $n \Vdash u_0 \sim u_1 \in [\![\Pi]\!](R, S)$ and $n' \leq n$: we need to show that $n' \Vdash u_0 \sim u_1 \in [\![\Pi]\!](R, S)$. Fixing $m \leq n'$ and v_0, v_1 which are in R at stage m, we have to observe that $\underline{\mathbf{app}}(u_0, v_0)$ and $\underline{\mathbf{app}}(u_1, v_1)$ are related at stage m in $S(v_0, v_1)$. This is immediate from our assumption, because $m \leq n' \leq n$.

□

Lemma 3.1.6. *If $R \in \mathbf{Rel}$ is monotone and each fiber $S(v_0, v_1)$ of a family $S \in \mathbf{Fam}$ is monotone for $n \Vdash v_0 \sim v_1 \in R$, then $[\![\Sigma]\!](R, S)$ is monotone.*

Proof. Suppose $n \Vdash u_0 \sim u_1 \in [\![\Sigma]\!](R, S)$ and $m \leq n$: we need to show that $m \Vdash u_0 \sim u_1 \in [\![\Sigma]\!](R, S)$.

1. To see that $m \Vdash \underline{\mathbf{fst}}(u_0) \sim \underline{\mathbf{fst}}(u_1) \in R$, observe that $n \Vdash \underline{\mathbf{fst}}(u_0) \sim \underline{\mathbf{fst}}(u_1) \in R$ and use the monotonicity of S.

2. To see that $m \Vdash \underline{\mathbf{fst}}(u_0) \sim \underline{\mathbf{fst}}(u_1) \in S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_1))$, observe that $n \Vdash \underline{\mathbf{fst}}(u_0) \sim \underline{\mathbf{fst}}(u_1) \in S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_1))$ and use the monotonicity of $S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_1))$. □

Lemma 3.1.7. *If $R \in \mathbf{Rel}$ is a PER, then $[\![\Box]\!](R)$ is a monotone PER.*

Proof. $[\![\Box]\!](R)$ is clearly monotone, because its definition discards the index.

1. *Symmetry.* Suppose that $n \Vdash u_0 \sim u_1 \in [\![\Box]\!](R)$; we need to see that $n \Vdash u_1 \sim u_0 \in [\![\Box]\!](R)$, which is to say that for all m, $m \Vdash \underline{\mathbf{open}}(u_1) \sim \underline{\mathbf{open}}(u_0) \in R$. By symmetry of R, it suffices to show that $m \Vdash \underline{\mathbf{open}}(u_0) \sim \underline{\mathbf{open}}(u_1) \in R$, which we have already assumed.

2. *Transitivity.* Analogous to symmetry. □

Lemma 3.1.8. *If $R \in \mathbf{Rel}$ is a monotone PER and $v_0, v_1 \in \mathbf{Val}$, then $[\![\mathsf{Id}]\!](R, v_0, v_1)$ is a monotone PER.*

Proof. $[\![\mathsf{Id}]\!](R, v_0, v_1)$ is clearly monotone as we have assumed that R is monotone.

1. *Symmetry.* There are two cases to consider here.

   a) Suppose that $n \Vdash \mathbf{refl}(u_0) \sim \mathbf{refl}(u_1) \in [\![\mathsf{Id}]\!](R, v_0, v_1)$; we need to see that $n \Vdash \mathbf{refl}(u_1) \sim \mathbf{refl}(u_0) \in [\![\mathsf{Id}]\!](R, v_0, v_1)$, which is to say $m \Vdash v_0 \sim u_0 \in R$, $m \Vdash u_1 \sim u_0 \in R$, and $m \Vdash u_1 \sim v_1 \in R$.
   We have by assumption $m \Vdash v_0 \sim u_0 \in R$, $m \Vdash u_0 \sim u_1 \in R$, and $m \Vdash u_1 \sim v_1 \in R$ so the result is immediate from the symmetry of R.

   b) Suppose instead that $n \Vdash \uparrow^{\mathbf{Id}(-,-,-)} e_0 \sim \uparrow^{\mathbf{Id}(-,-,-)} e_1 \in [\![\mathsf{Id}]\!](R, v_0, v_1)$ and so $e_0 \sim e_1 \in \mathcal{N}e$. We wish to show that $n \Vdash \uparrow^{\mathbf{Id}(-,-,-)} e_1 \sim \uparrow^{\mathbf{Id}(-,-,-)} e_0 \in [\![\mathsf{Id}]\!](R, v_0, v_1)$ holds but this is immediate as $\mathcal{N}e$ is a PER.

2. *Transitivity.* Analogous to symmetry. □

Defining the type hierarchy

We begin by defining the individual closure of a type system $\sigma \in \mathbf{Sys}$ under each of the connectives of our type theory, as well as under the neutral types. We present these definitions as inference rules.

Each rule defines the closure of a type-system under a particular connective.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad \sigma \models_n R \gg B_0 \sim B_1 \downarrow S}{\mathbf{Pi}[\sigma] \models_n \Pi(A_0, B_0) \sim \Pi(A_1, B_1) \downarrow [\![\Pi]\!](R, S)} \qquad \frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad \sigma \models_n R \gg B_0 \sim B_1 \downarrow S}{\mathbf{Sg}[\sigma] \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow [\![\Sigma]\!](R, S)}$$

$$\frac{R : \mathbf{SFam} \qquad \forall m.\ \sigma \models_m A_0 \sim A_1 \downarrow R(m) \qquad S = \{(n, u_0, u_1) \mid n \Vdash u_0 \sim u_1 \in R(n)\}}{\mathbf{Box}[\sigma] \models_n \Box A_0 \sim \Box A_1 \downarrow [\![\Box]\!](S)}$$

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad n \Vdash v_0 \sim u_0 \in R \qquad n \Vdash v_1 \sim u_1 \in R}{\mathbf{Id}[\sigma] \models_n \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow [\![\mathrm{Id}]\!](R, u_0, u_1)}$$

$$\frac{e_0 \sim e_1 \in \mathcal{N}e \qquad R = \{(m, \uparrow^{B_0} e_0, \uparrow^{B_1} e_1) \mid e_0 \sim e_1 \in \mathcal{N}e\}}{\mathbf{Ne} \models_n \uparrow^{A_0} e_0 \sim \uparrow^{A_1} e_1 \downarrow R} \qquad \frac{}{\mathbf{Nat} \models_n \mathbf{nat} \sim \mathbf{nat} \downarrow [\![\mathbb{N}]\!]}$$

Next, we define the hierarchy of universes by iterating the closure of a type system under connectives up to the infinite ordinal ω, letting α range over $\mathbb{N} \cup \{\omega\}$:

$$\frac{j < \alpha}{\mathbf{Univ}_\alpha \models_n \mathbf{U}_j \sim \mathbf{U}_j \downarrow \{(m, A_0, A_1) \mid \tau_j \models_m A_0 \sim A_1\}}$$

$$\mathbf{Types}_\alpha[\sigma] = \mathbf{Pi}[\sigma] \vee \mathbf{Sg}[\sigma] \vee \mathbf{Box}[\sigma] \vee \mathbf{Id}[\sigma] \vee \mathbf{Nat} \vee \mathbf{Univ}_\alpha \vee \mathbf{Ne} \qquad \tau_\alpha = \mu\sigma.\ \mathbf{Types}_\alpha[\sigma]$$

The ultimate type system τ_ω has types at every level, including all universes $\mathbf{U}_i$ of finite level.

3.2 *Properties of the PER model*

For clarity, and because we shall so frequently make use of this fact in the following proofs, let us now take a moment to state the universal property of μ.

Theorem 3.2.1 (Universal Property of a Least Fixed Point). *If μF is the least fixed point of $F : L \to L$ then for any $x : L$ such that $F(x) \leq x$ we must have $\mu F \leq x$.*

Remark 3.2.2. If $F(x) \leq x$ we shall call x a pre-fixed point of F.

Remark 3.2.3. In what follows we will use α, β, γ, to denote either some natural number n or ω. Recall that τ_α is defined for all of these values and all the properties we wish to show must be proven for both n and ω.

Lemma 3.2.4 (Determinism). *For any α, τ_α is deterministic. That is, if $\tau_\alpha \models_n A \sim B \downarrow R$ and $\tau_\alpha \models_n A \sim B \downarrow R'$, then $R = R'$.*

Proof. This proof proceeds by showing that the following σ is pre-fixed point of $\mathbf{Types}_\alpha[-]$:

$$\frac{\tau_\alpha \models_n A \sim B \downarrow R \qquad \forall R'.\ \tau_\alpha \models_n A \sim B \downarrow R' \implies R = R'}{\sigma \models_n A \sim B \downarrow R}$$

Once this has been established, we then conclude that $\tau_\alpha \leq \sigma$ which in turn implies that τ_α must be deterministic. As usual, we exhibit only the cases pertaining to non-standard extensions of Martin-Löf Type Theory.

Supposing that we have $\mathbf{Types}_\alpha[\sigma] \models_n A \sim B \downarrow R$, we wish to show that $\sigma \models_n A \sim B \downarrow R$ holds as well. We proceed by case:

Case.

$$\mathbf{Univ}_\alpha \models_n \mathbf{U}_i \sim \mathbf{U}_i \downarrow R \text{ where } i < \alpha \text{ and } R = \{(m, A_0, A_1) \mid \tau_i \models_m A_0 \sim A_1\}$$

First, we need to show that $\tau_\alpha \models_n \mathbf{U}_i \sim \mathbf{U}_i \downarrow R$, but this follows immediately from our assumption, which is one of the generators of the type system closure. Next, supposing that $\tau_\alpha \models_n \mathbf{U}_i \sim \mathbf{U}_i \downarrow S$, we need to verify that $R = S$. But by inverting the type system closure, we must have $\mathbf{Univ}_\alpha \models_n \mathbf{U}_i \sim \mathbf{U}_i \downarrow S$, from which we conclude $R = S$.

Case.

$$\frac{\forall m.\ \sigma \models_m A_0 \sim A_1 \downarrow R(m) \qquad S = \{(n, u_0, u_1) \mid n \Vdash u_0 \sim u_1 \in R(n)\}}{\mathbf{Box}[\sigma] \models_n \Box A_0 \sim \Box A_1 \downarrow [\![\Box]\!](S)}$$

Because $\sigma \le \tau_\alpha$, we can see that $\mathbf{Box}[\tau_\alpha] \models_n \Box A_0 \sim \Box A_1 \downarrow [\![\Box]\!](S)$ and therefore $\tau_\alpha \models_n \Box A_0 \sim \Box A_1 \downarrow [\![\Box]\!](S)$. Fixing $T \in \mathbf{Rel}$ such that $\tau_\alpha \models_n \Box A_0 \sim \Box A_1 \downarrow T$, we need to verify that $T = [\![\Box]\!](S)$. By inverting the type system closure, we have $\mathbf{Box}[\tau_\alpha] \models_n \Box A_0 \sim \Box B_0 \downarrow T$; by definition, this means that we have some family of relations $R' \in \mathbf{Rel}^{\mathbb{P}}$ where $\tau_\alpha \models_m A_0 \sim A_1 \downarrow R'(m)$ for each m, and moreover $T = [\![\Box]\!](\{(n, u_0, u_1) \mid n \Vdash u_0 \sim u_1 \in R'(n)\})$.

Therefore, it remains to see that $R' = R$; but this is immediate from the fact that both are contained in the type system σ: unfolding, we have both $\tau_\alpha \models_m A_0 \sim A_1 \downarrow R(m)$ and for all $R'' \in \mathbf{Rel}$, if $\tau_\alpha \models_m A_0 \sim A_1 \downarrow R''$ then $R(m) = R''$. Therefore, to see that $R'(m) = R(m)$, we choose $R'' = R'(m)$ and use the fact that $\tau_\alpha \models_m A_0 \sim A_1 \downarrow R'(m)$. □

A number of properties of this type system must be established simultaneously because of interdependency.

Lemma 3.2.5. *For any α, the following properties hold.*

1. *If $\tau_\alpha \models_n A \sim B \downarrow R$ then $\tau_\alpha \models_n B \sim A \downarrow R$.*
2. *If $\tau_\alpha \models_n A \sim B \downarrow R$ and $\tau_\alpha \models_n B \sim C \downarrow R$, then $\tau_\alpha \models_n A \sim C \downarrow R$.*
3. *If $\tau_\alpha \models_n A \sim B \downarrow R$ and $m \le n$, then $\tau_\alpha \models_m A \sim B \downarrow R$.*
4. *If $\tau_\alpha \models_n A \sim B \downarrow R$ then R is a monotone PER.*

Proof. We prove these statements by strong induction on α. This induction on the level is necessary in the case of $\mathbf{Univ}_\alpha$. Here, for instance, in order to show that the relation on terms is monotone we need to know that the relation on types is monotone for all $i < \alpha$. Similarly with symmetry and transitivity.

Let us assume therefore that for any $i < \alpha$ the following facts hold:

1. If $\tau_i \models_n A \sim B \downarrow R$ then $\tau_i \models_n B \sim A \downarrow R$.
2. If $\tau_i \models_n A \sim B \downarrow R$ and $\tau_i \models_n B \sim C \downarrow R$, then $\tau_i \models_n A \sim C \downarrow R$.
3. If $\tau_i \models_n A \sim B \downarrow R$ and $m \le n$, then $\tau_i \models_m A \sim B \downarrow R$.
4. If $\tau_i \models_n A \sim B \downarrow R$ then R is a monotone PER.

We note that the above makes $(\tau_i \models_{(-)} - \sim -)$ a monotone PER.

We now turn to showing that these facts hold for α, all of which must be established simultaneously. This is done by showing the following $\sigma \in \mathbf{Sys}$ to be a pre-fixed point:

$$\frac{\begin{array}{l} R \text{ is a monotone PER} \\ \forall m \leq n.\ \tau_\alpha \models_m A \sim B \downarrow R \\ \tau_\alpha \models_n B \sim A \downarrow R \\ \forall C, S.\ \tau_\alpha \models_n B \sim C \downarrow S \implies \tau_\alpha \models_n A \sim C \downarrow S \wedge (R = S) \\ \forall C, S.\ \tau_\alpha \models_n C \sim A \downarrow S \implies \tau_\alpha \models_n C \sim B \downarrow S \wedge (R = S) \end{array}}{\sigma \models_n A \sim B \downarrow R}$$

Supposing that $\mathbf{Types}_\alpha[\sigma] \models_n A \sim B \downarrow R$, we must show that $\sigma \models_n A \sim B \downarrow R$. We proceed by case.

Case.

$$\mathbf{Univ}_\alpha \models_n \mathbf{U}_i \sim \mathbf{U}_i \downarrow R \text{ where } i < \alpha \text{ and } R = \{(m, A_0, A_1) \mid \tau_i \models_m A_0 \sim A_1\}$$

First, we observe that for any $m \leq n$, we also have $\mathbf{Univ}_\alpha \models_m \mathbf{U}_i \sim \mathbf{U}_i \downarrow R$ and thence $\tau_\alpha \models_m \mathbf{U}_i \sim \mathbf{U}_i \downarrow R$. Symmetry is trivial, because we have the same type on both sides. We need to show both directions of the generalized transitivity.

- Suppose that $\tau_\alpha \models_n \mathbf{U}_i \sim C \downarrow S$; we need to verify that $R = S$. By inversion, we must have $C = \mathbf{U}_i$ and moreover $R = S$.
- Suppose that $\tau_\alpha \models_n C \sim \mathbf{U}_i \downarrow S$; we need to verify that $R = S$. By inversion, we must have $C = \mathbf{U}_i$ and moreover $R = S$.

Finally, we must show that R is a monotone PER; by the definition of R above, it it suffices to recall that $(\tau_i \models_{(-)} - \sim -)$ is a monotone PER.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad \sigma \models_n R \gg B_0 \sim B_1 \downarrow S}{\mathbf{Pi}[\sigma] \models_n \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1) \downarrow [\![\Pi]\!](R, S)}$$

Before establishing the main properties of the dependent function connective, we first observe that for any $m \Vdash a_0 \sim a_1 \in R$, the relations $S(a_0, a_1)$, $S(a_1, a_1)$ and $S(a_1, a_0)$ are equal fibers of S. To achieve this, we execute a brutal *power move* described in Angiuli [Ang19]. Because R is a PER, we can conclude the following:

$$\sigma \models_{m'} B_0[a_0] \sim B_1[a_1] \downarrow S(a_0, a_1) \tag{3.1}$$
$$\sigma \models_{m'} B_0[a_1] \sim B_1[a_0] \downarrow S(a_1, a_0) \tag{3.2}$$
$$\sigma \models_{m'} B_0[a_1] \sim B_1[a_1] \downarrow S(a_1, a_1) \tag{3.3}$$

Unfolding (3.1,3.2), we obtain the following symmetric instances:

$$\tau_\alpha \models_{m'} B_1[a_1] \sim B_0[a_0] \downarrow S(a_0, a_1) \tag{3.4}$$
$$\tau_\alpha \models_{m'} B_1[a_0] \sim B_0[a_1] \downarrow S(a_1, a_0) \tag{3.5}$$

Unfolding (3.3) we have the following generalized transitivities:

$$\forall C, T.\ \tau_\alpha \models_{m'} B_1[a_1] \sim C \downarrow T \implies \tau_\alpha \models_{m'} B_0[a_1] \sim C \downarrow T \wedge S(a_1, a_1) = T \tag{3.6}$$
$$\forall C, T.\ \tau_\alpha \models_{m'} C \sim B_0[a_1] \downarrow T \implies \tau_\alpha \models_{m'} C \sim B_1[a_1] \downarrow T \wedge S(a_1, a_1) = T \tag{3.7}$$

Instantiating (3.6) with (3.4) we obtain $S(a_1, a_1) = S(a_0, a_1)$; instantiating (3.7) with (3.5) we further obtain $S(a_1, a_1) = S(a_1, a_0)$. Therefore, $S(a_0, a_1) = S(a_1, a_0)$.

1. $[\![\Pi]\!](R, S)$ *is a monotone PER.* Monotonicity is given by 3.1.5; but we need to show that it is symmetric and transitive.
   a) *Symmetry.* Suppose that $m \Vdash v_0 \sim v_1 \in [\![\Pi]\!](R, S)$; we need to show that $m \Vdash v_1 \sim v_0 \in [\![\Pi]\!](R, S)$. Fixing $m' \leq m$ and $m' \Vdash a_0 \sim a_1 \in R$, we need to show that $m' \Vdash \underline{\mathbf{app}}(u_1, a_0) \sim \underline{\mathbf{app}}(u_0, a_1) \in S(a_0, a_1)$. We note by assumption that $m' \Vdash a_1 \sim a_0 \in R$ and therefore $m' \Vdash \underline{\mathbf{app}}(u_1, a_0) \sim \underline{\mathbf{app}}(u_0, a_1) \in S(a_1, a_0)$. Therefore, it would suffice to observe that $S(a_0, a_1)_{m'} = S(a_1, a_0)_{m'}$, which we have above.
   b) *Transitivity.* Suppose that $m \Vdash u_0 \sim u_1 \in [\![\Pi]\!](R, S)$ and $m \Vdash u_1 \sim u_2 \in [\![\Pi]\!](R, S)$; we need to show that $m \Vdash u_0 \sim u_2 \in [\![\Pi]\!](R, S)$. Fixing $m' \leq m$ and $R \Vdash a_0 \sim a_1 \in m'$, we need to show that $m' \Vdash \underline{\mathbf{app}}(u_0, a_0) \sim \underline{\mathbf{app}}(u_2, a_1) \in S(a_0, a_1)$. We obtain the following from our assumptions:

$$m' \Vdash \underline{\mathbf{app}}(u_0, a_0) \sim \underline{\mathbf{app}}(u_1, a_1) \in S(a_0, a_1) \tag{3.8}$$

$$m' \Vdash \underline{\mathbf{app}}(u_0, a_1) \sim \underline{\mathbf{app}}(u_1, a_0) \in S(a_1, a_0) \tag{3.9}$$

$$m' \Vdash \underline{\mathbf{app}}(u_1, a_0) \sim \underline{\mathbf{app}}(u_2, a_1) \in S(a_0, a_1) \tag{3.10}$$

   Using (3.9,3.10) and the fact that S is transitive, it suffices to observe that $S(a_0, a_1) = S(a_1, a_0)$, which we have already shown.
2. *For all $m \leq n$, we have $\tau_\alpha \models_m \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1) \downarrow [\![\Pi]\!](R, S)$.* Fixing $m \leq n$, we need to show two things.
   a) $\tau_\alpha \models_m A_0 \sim A_1 \downarrow R$ can be obtained from our assumption that $\sigma \models_n A_0 \sim A_1 \downarrow R$.
   b) To see that $\tau_\alpha \models_m R \gg B_0 \sim B_1 \downarrow S$ holds, we fix $m' \leq m$ and $m' \Vdash a_0 \sim a_1 \in R$, and need to verify that $\tau_\alpha \models_{m'} B_0[a_0] \sim B_1[a_1] \downarrow S(a_0, a_1)$. Instantiating our assumption $\sigma \models_n R \gg B_0 \sim B_1 \downarrow S$ with $m' \leq m \leq n$, we obtain $\sigma \models_{m'} B_0[a_0] \sim B_1[a_1] \downarrow S(a_0, a_1)$, whence $\tau_\alpha \models_{m'} B_0[a_0] \sim B_1[a_1] \downarrow S(a_0, a_1)$.
3. $\tau_\alpha \models_n \mathbf{\Pi}(A_1, B_1) \sim \mathbf{\Pi}(A_0, B_0) \downarrow [\![\Pi]\!](R, S)$.
   a) $\tau_\alpha \models_m A_1 \sim A_0 \downarrow R$ is obtained from our assumption that $\sigma \models_n A_0 \sim A_1 \downarrow R$.
   b) To see that $\tau_\alpha \models_m R \gg B_1 \sim B_0 \downarrow S$ holds, we fix $m \leq n$ and $m \Vdash a_0 \sim a_1 \in R$, needing to verify that $\tau_\alpha \models_m B_1[a_0] \sim B_0[a_1] \downarrow S(a_0, a_1)$. We have already seen that $S(a_0, a_1) = S(a_1, a_0)$, so it suffices to show that $\tau_\alpha \models_m B_1[a_0] \sim B_0[a_1] \downarrow S(a_1, a_0)$. But this is one of the symmetric instances of our assumption $\sigma \models_n R \gg B_0 \sim B_1 \downarrow S$, considering $m \Vdash a_1 \sim a_0 \in R$.
4. *If $\tau_\alpha \models_n \mathbf{\Pi}(A_1, B_1) \sim C \downarrow T$, then $\tau_\alpha \models_n \mathbf{\Pi}(A_0, B_0) \sim C \downarrow T$ and moreover $T = [\![\Pi]\!](R, S)$.* By inversion, we have $C = \mathbf{\Pi}(A_2, B_2)$ and $T = [\![\Pi]\!](U, V)$ such that $\tau_\alpha \models_n A_1 \sim A_2 \downarrow U$ and $\tau_\alpha \models_n U \gg B_1 \sim B_2 \downarrow V$. We need to verify that $\tau_\alpha \models_n \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_2, B_2) \downarrow [\![\Pi]\!](U, V)$.
   a) To see that $\tau_\alpha \models_n A_0 \sim A_2 \downarrow U$, we recall that our assumption $\sigma \models_n A_0 \sim A_1 \downarrow R$ contains a generalized transitivity which, when instantiated with $\tau_\alpha \models_n A_1 \sim A_2 \downarrow U$, obtains both our goal $\tau_\alpha \models_n A_0 \sim A_2 \downarrow U$ and moreover $R = U$.
   b) Now we have to show that $\tau_\alpha \models_n R \gg B_0 \sim B_2 \downarrow V$. Fixing $m \leq n$ and $m \Vdash a_0 \sim a_1 \in R$, we need to verify that $\tau_\alpha \models_m B_0[a_0] \sim B_2[a_1] \downarrow V(a_0, a_1)$. Instantiating one of our hypotheses with $m \Vdash a_1 \sim a_1 \in R$, we have:

$$\tau_\alpha \models_m B_1[a_1] \sim B_2[a_1] \downarrow V(a_1, a_1) \tag{3.11}$$

   By assumption, we obtain $\sigma \models_m B_0[a_0] \sim B_1[a_1] \downarrow S(a_0, a_1)$, and using its generalized transitivity at (3.11), we obtain $\tau_\alpha \models_m B_0[a_0] \sim B_2[a_1] \downarrow V(a_1, a_1)$ such that $V(a_1, a_1) = S(a_0, a_1)$. It remains only to see that $V(a_1, a_1) = V(a_0, a_1)$, but we have already seen that this is the case.

c) It remains only to observe that $S = V$; but we had both $V(a_1, a_1) = V(a_0, a_1)$ and $V(a_1, a_1) = S(a_0, a_1)$.

5. *If $\tau_\alpha \models_n C \sim \mathbf{\Pi}(A_0, B_0) \downarrow T$, then $\tau_\alpha \models_n C \sim \mathbf{\Pi}(A_1, B_1) \downarrow T$ and moreover $T = [\![\Pi]\!](R, S)$.* This is symmetric to the previous case.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad \sigma \models_n R \gg B_0 \sim B_1 \downarrow S}{\mathbf{Sg}[\sigma] \models_n \mathbf{\Sigma}(A_0, B_0) \sim \mathbf{\Sigma}(A_1, B_1) \downarrow [\![\Sigma]\!](R, S)}$$

We show only that $[\![\Sigma]\!](R, S)$ is a monotone PER; the other properties are exactly as in the case for **Pi**.

1. *Monotonicity.* By Lemma 3.1.6 it suffices to show that both R and S are monotone, both of which are obtained by assumption.
2. *Symmetry.* Suppose $m \Vdash u_0 \sim u_1 \in [\![\Sigma]\!](R, S)$; we need to show that $m \Vdash u_1 \sim u_0 \in [\![\Sigma]\!](R, S)$.
   a) We obtain $m \Vdash \underline{\mathbf{fst}}(u_1) \sim \underline{\mathbf{fst}}(u_0) \in R$ from $m \Vdash \underline{\mathbf{fst}}(u_0) \sim \underline{\mathbf{fst}}(u_1) \in R$ using our induction hypothesis.
   b) Next, we need to see that $m \Vdash \underline{\mathbf{snd}}(u_1) \sim \underline{\mathbf{snd}}(u_0) \in S(\underline{\mathbf{fst}}(u_1), \underline{\mathbf{fst}}(u_0))$. We obtain $m \Vdash \underline{\mathbf{snd}}(u_1) \sim \underline{\mathbf{snd}}(u_0) \in S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_1))$ from $m \Vdash \underline{\mathbf{snd}}(u_0) \sim \underline{\mathbf{snd}}(u_1) \in S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_1))$ using our induction hypothesis, so it suffices to see observe that $S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_1)) = S(\underline{\mathbf{fst}}(u_1), \underline{\mathbf{fst}}(u_0))$, which we have already proved.
3. *Transitivity.* Suppose $m \Vdash u_0 \sim u_1 \in [\![\Sigma]\!](R, S)$ and $m \Vdash u_1 \sim u_2 \in [\![\Sigma]\!](R, S)$; we need to show that $m \Vdash u_0 \sim u_2 \in [\![\Sigma]\!](R, S)$.
   a) We obtain $m \Vdash \underline{\mathbf{fst}}(u_0) \sim \underline{\mathbf{fst}}(u_2) \in R$ using the transitivity of R, which we have assumed.
   b) It remains to show that $m \Vdash \underline{\mathbf{snd}}(u_0) \sim \underline{\mathbf{snd}}(u_2) \in S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_2))$. By transitivity of S, it suffices to show that $S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_1)) = S(\underline{\mathbf{fst}}(u_1), \underline{\mathbf{fst}}(u_2)) = S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_2))$. But we have already observed that this is entailed by $m \Vdash \underline{\mathbf{fst}}(u_0) \sim \underline{\mathbf{fst}}(u_1) \in R$ and $m \Vdash \underline{\mathbf{fst}}(u_1) \sim \underline{\mathbf{fst}}(u_2) \in R$.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad n \Vdash v_0 \sim u_0 \in R \qquad n \Vdash v_1 \sim u_1 \in R}{\mathbf{Id}[\sigma] \models_n \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)}$$

1. *$[\![\mathsf{Id}]\!](R, u_0, u_1)$ is a monotone PER.* By Lemma 3.1.8.
2. *For $n' \leq n$ we have $\tau_\alpha \models_{n'} \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)$.* Observe that we have $\sigma \models_n A_0 \sim B_0 \downarrow R$ and therefore $\tau_\alpha \models_{n'} A_0 \sim B_0 \downarrow R$ along with $n \Vdash u_0 \sim v_0 \in R$, and $n \Vdash u_1 \sim v_1 \in R$. Our goal is immediate as R must be monotone.
3. *We have $\tau_\alpha \models_n \mathbf{Id}(A_1, u_0, u_1) \sim \mathbf{Id}(A_0, v_0, v_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)$.* Observe that we have $\sigma \models_n A_0 \sim A_1 \downarrow R$ and therefore we know that R is a monone PER as well as $\tau_\alpha \models_n A_1 \sim A_0 \downarrow R$. As noted above, we have $n \Vdash u_0 \sim v_0 \in R$ and $n \Vdash u_1 \sim v_1 \in R$ so the symmetry of R tells us that $n \Vdash v_0 \sim u_0 \in R$ and $n \Vdash v_1 \sim u_1 \in R$. Again, because R is a monotone PER we must have that $[\![\mathsf{Id}]\!](R, u_0, u_1) = [\![\mathsf{Id}]\!](R, v_0, v_1)$. Therefore, we have $\tau_\alpha \models_n \mathbf{Id}(A_1, u_0, u_1) \sim \mathbf{Id}(A_0, v_0, v_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)$ as required.
4. *If $\tau_\alpha \models_n \mathbf{Id}(A_1, u_0, u_1) \sim C \downarrow T$, then $\tau_\alpha \models_n \mathbf{Id}(A_0, v_0, v_1) \sim C \downarrow T$ and moreover $T = [\![\mathsf{Id}]\!](R, u_0, u_1)$.* By inversion, we have $C = \mathbf{Id}(A_2, w_0, w_1)$ and $T = [\![\mathsf{Id}]\!](S, w_0, w_1)$ for some S such that $\tau_\alpha \models_n A_1 \sim A_2 \downarrow S$, $n \Vdash u_0 \sim w_0 \in S$ and $n \Vdash u_1 \sim w_1 \in S$. Let us first observe that by induction hypothesis that we have $\tau_\alpha \models_{A_0} A_2 \sim S$ and $S = R$. Therefore, we may conclude that $n \Vdash v_0 \sim w_0 \in R$ and $n \Vdash v_1 \sim w_1 \in R$ as $R = S$ and R is a monotone PER. This also tells us that $T = [\![\mathsf{Id}]\!](R, u_0, u_1)$.

   Therefore, we have $\tau_\alpha \models_n \mathbf{Id}(A_0, v_0, v_1) \sim C \downarrow T$ as required.

5. *If* $\tau_\alpha \models_n C \sim \mathbf{Id}(A_0, v_0, v_1) \downarrow T$, *then* $\tau_\alpha \models_n C \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow T$ *and moreover* $T = [\![\mathsf{Id}]\!](R, u_0, u_1)$. Identical to the above.

Case.

$$\frac{\forall m.\ \sigma \models_m A_0 \sim A_1 \downarrow R(m) \qquad S = \{(n, u_0, u_1) \mid n \Vdash u_0 \sim u_1 \in R(n)\}}{\mathbf{Box}[\sigma] \models_n \Box A_0 \sim \Box A_1 \downarrow [\![\Box]\!](S)}$$

1. $[\![\Box]\!](S)$ *is a monotone PER.* By Lemma 3.1.7.
2. *For* $n' \leq n$ *we have* $\tau_\alpha \models_{n'} \Box A_0 \sim \Box A_1 \downarrow [\![\Box]\!](S)$. Observe that we have $\sigma \models_{n'} A \sim B \downarrow R(n')$, and thence $\sigma \models_{n'} A \sim B \downarrow R(n')$. Our goal is immediate.
3. *We have* $\tau_\alpha \models_n \Box A_1 \sim \Box A_0 \downarrow [\![\Box]\!](S)$. Observe that we have $\sigma \models_m A_0 \sim A_1 \downarrow R(m)$ for all m, and therefore also $\tau_\alpha \models_m A_1 \sim A_0 \downarrow R(m)$, from which we conclude $\tau_\alpha \models_n \Box A_1 \sim \Box A_0 \downarrow [\![\Box]\!](S)$.
4. *If* $\tau_\alpha \models_n \Box A_1 \sim C \downarrow T$, *then* $\tau_\alpha \models_n \Box A_0 \sim C \downarrow T$ *and moreover* $T = [\![\Box]\!](S)$. By inversion, we have $C = \Box A_2$ and $T = [\![\Box]\!](\{(n, u_0, u_1) \mid n \Vdash u_0 \sim u_1 \in U(n)\})$ for some $U \in \mathbf{Rel}^{\mathbb{P}}$, such that for all m, we have $\tau_\alpha \models_m A_1 \sim A_2 \downarrow U(m)$.
   a) We need to show that $\tau_\alpha \models_n \Box A_0 \sim \Box A_2 \downarrow [\![\Box]\!](\{(n, u_0, u_1) \mid n \Vdash u_0 \sim u_1 \in U(n)\})$. It suffices to show that for all m, we have $\tau_\alpha \models_n A_0 \sim A_2 \downarrow U(m)$. Because both $\sigma \models_m A_0 \sim A_1 \downarrow R(m)$ and $\tau_\alpha \models_m A_1 \sim A_2 \downarrow U(m)$, we have by generalized transitivity both $\tau_\alpha \models_m A_0 \sim A_2 \downarrow U(m)$ and $U(m) = R(m)$ for all m.
   b) We have already observed that $U = R$, so clearly $T = [\![\Box]\!](S)$.
5. *If* $\tau_\alpha \models_n C \sim \Box A_0 \downarrow T$, *then* $\tau_\alpha \models_n C \sim \Box A_1 \downarrow T$ *and moreover* $T = [\![\Box]\!](S)$. Identical to the above.

$\square$

Lemma 3.2.6. *If* $e_0 \sim e_1 \in \mathcal{Ne}$ *then* $\tau_\alpha \models_n \uparrow^{\mathrm{U}_i} e_0 \sim \uparrow^{\mathrm{U}_i} e_1$ *for any* $i < \alpha$.

Proof. We have $\mathbf{Ne} \models_n \uparrow^{\mathrm{U}_i} e_0 \sim \uparrow^{\mathrm{U}_i} e_1$. $\square$

Lemma 3.2.7 (Compatibility). *Each* τ_α *is compatible and valued in compatible PERs (recall Definition 3.1.4). The partial equivalence relation given by* $\tau_\alpha \models_m - \sim -$ *is compatible for types and if* $\tau_\alpha \models_m A_0 \sim A_1 \downarrow R$ *then* R *is compatible for* (A_0, A_1).

Proof. We proceed by strong induction on α, and then show that the following $\sigma \in \mathbf{Sys}$ is a pre-fixed point of each $\mathbf{Types}_\alpha[-]$:

$$\frac{A_0 \sim A_1 \in \mathcal{Ty} \qquad \tau_\alpha \models_m A_0 \sim A_1 \downarrow R \qquad R \text{ is compatible}}{\sigma \models_m A_0 \sim A_1 \downarrow R}$$

Supposing that $\mathbf{Types}_\alpha[\sigma] \models_n A_0 \sim A_1 \downarrow R$, we establish $\sigma \models_n A_0 \sim A_1 \downarrow R$ by case.

Case.

$$\mathbf{Univ}_\alpha \models_n \mathbf{U}_i \sim \mathbf{U}_i \downarrow R \text{ where } i < \alpha \text{ and } R = \{(m, A_0, A_1) \mid \tau_i \models_m A_0 \sim A_1\}$$

We only need to observe that R is compatible.

1. Suppose $e_0 \sim e_1 \in \mathcal{Ne}$; by Lemma 3.2.6 we have $\tau_i \models_n \uparrow^{\mathrm{U}_i} e_0 \sim \uparrow^{\mathrm{U}_i} e_1$.
2. Suppose $\tau_i \models_n A_0 \sim A_1$; we observe that $\downarrow^{\mathrm{U}_i} A_0 \sim \downarrow^{\mathrm{U}_i} A_1 \in \mathcal{Nf}$ follows from $A_0 \sim A_1 \in \mathcal{Ty}$, which we obtain from our induction hypothesis at $i < \alpha$.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad \sigma \models_n R \gg B_0 \sim B_1 \downarrow S}{\mathbf{Pi}[\sigma] \models_n \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1) \downarrow [\![\Pi]\!](R, S)}$$

1. First, we check that $\mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1) \in \mathcal{Ty}$. It suffices to check the following:
   a) $A_0 \sim A_1 \in \mathcal{Ty}$, which is obtained $\sigma \models_n A_0 \sim A_1 \downarrow R$.
   b) $\tau_\alpha \models_n \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1)$ follows from our induction hypotheses.
   c) For all k, $B_0[\uparrow^{A_0} \mathbf{var}_k] \sim B_1[\uparrow^{A_1} \mathbf{var}_k] \in \mathcal{Ty}$. To see that this holds, we observe because R is compatible with (A_0, A_1), we have $n \Vdash \uparrow^{A_0} \mathbf{var}_k \sim \uparrow^{A_1} \mathbf{var}_k \in R$ and therefore $\sigma \models_n B_0[\uparrow^{A_0} \mathbf{var}_k] \sim B_1[\uparrow^{A_1} \mathbf{var}_k]$; from this, we obtain $B_0[\uparrow^{A_0} \mathbf{var}_k] \sim B_1[\uparrow^{A_1} \mathbf{var}_k] \in \mathcal{Ty}$.
2. Next, we check that $[\![\Pi]\!](R, S)$ is compatible with $(\mathbf{\Pi}(A_0, B_0), \mathbf{\Pi}(A_1, B_1))$.
   a) Suppose $e_0 \sim e_1 \in \mathcal{Ne}$; we need to show that $n \Vdash \uparrow^{\mathbf{\Pi}(A_0,B_0)} e_0 \sim \uparrow^{\mathbf{\Pi}(A_1,B_1)} e_1 \in [\![\Pi]\!](R, S)$. Fixing $m \leq n$ and $m \Vdash v_0 \sim v_1 \in R$, we must verify that $m \Vdash \underline{\mathbf{app}}(\uparrow^{\mathbf{\Pi}(A_0,B_0)} e_0, v_0) \sim \underline{\mathbf{app}}(\uparrow^{\mathbf{\Pi}(A_1,B_1)} e_1, v_1) \in S(v_0, v_1)$, which reduces to showing $m \Vdash \uparrow^{B_0[v_0]} e_0.\mathbf{app}(\downarrow^{A_0} v_0) \sim \uparrow^{B_1[v_1]} e_1.\mathbf{app}(\downarrow^{A_1} v_1) \in S(v_0, v_1)$. By induction, the fiber $S(v_0, v_1)$ is compatible with $(B_0[v_0], B_1[v_1])$, so it would suffice to know that $e_0.\mathbf{app}(\downarrow^{A_0} v_0) \sim e_1.\mathbf{app}(\downarrow^{A_1} v_0) \in \mathcal{Ne}$. This in turn follows from $e_0 \sim e_1 \in \mathcal{Ne}$ (which we have assumed), and $\downarrow^{A_0} v_0 \sim \downarrow^{A_1} v_1 \in \mathcal{Nf}$, which we obtain from the compatibility of R with (A_0, a_1) and our assumption $n \Vdash v_0 \sim v_1 \in R$.
   b) Suppose $n \Vdash u_0 \sim u_1 \in [\![\Pi]\!](R, S)$; we need to show that $\downarrow^{\mathbf{\Pi}(A_0,B_0)} u_0 \sim \downarrow^{\mathbf{\Pi}(A_1,B_1)} u_1 \in \mathcal{Nf}$. It suffices to show that for all k, $\downarrow^{B_0[\uparrow^{A_0}\mathbf{var}_k]} \underline{\mathbf{app}}(u_0, \uparrow^{A_0} \mathbf{var}_k) \sim \downarrow^{B_1[\uparrow^{A_1}\mathbf{var}_k]} \underline{\mathbf{app}}(u_1, \uparrow^{A_1} \mathbf{var}_k) \in \mathcal{Nf}$. First, observe that this would follow if we could show that $S(\uparrow^{A_0} \mathbf{var}_k, \uparrow^{A_1} \mathbf{var}_k)$ were compatible with $(B_0[\uparrow^{A_0} \mathbf{var}_k], B_1[\uparrow^{A_1} \mathbf{var}_k])$; this we can obtain from $n \Vdash \uparrow^{A_0} \mathbf{var}_k \sim \uparrow^{A_1} \mathbf{var}_k \in R$, which follows from the compatiblity of R with (A_0, A_1), and the fact that $\mathbf{var}_k \sim \mathbf{var}_k \in \mathcal{Ne}$.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad \sigma \models_n R \gg B_0 \sim B_1 \downarrow S}{\mathrm{Sg}[\sigma] \models_n \mathbf{\Sigma}(A_0, B_0) \sim \mathbf{\Sigma}(A_1, B_1) \downarrow [\![\Sigma]\!](R, S)}$$

1. We observe that $\mathbf{\Sigma}(A_0, B_0) \sim \mathbf{\Sigma}(A_1, B_1) \in \mathcal{Ty}$ in the exact same way that we did for dependent function types above.
2. $\tau_\alpha \models_n \mathbf{\Sigma}(A_0, B_0) \sim \mathbf{\Sigma}(A_1, B_1)$ follows from our induction hypotheses.
3. We check that $[\![\Sigma]\!](R, S)$ is compatible with $(\mathbf{\Sigma}(A_0, B_0), \mathbf{\Sigma}(A_1, B_1))$:
   a) Suppose that $e_0 \sim e_1 \in \mathcal{Ne}$; we need to show that $n \Vdash \uparrow^{\mathbf{\Sigma}(A_0,B_0)} e_0 \sim \uparrow^{\mathbf{\Sigma}(A_1,B_1)} e_1 \in [\![\Sigma]\!](R, S)$.
      i. We have to check $n \Vdash \underline{\mathbf{fst}}(\uparrow^{\mathbf{\Sigma}(A_0,B_0)} e_0) \sim \underline{\mathbf{fst}}(\uparrow^{\mathbf{\Sigma}(A_1,B_1)} e_1) \in R$, which is the same as to say, $n \Vdash \uparrow^{A_0} e_0.\mathbf{fst} \sim \uparrow^{A_1} e_1.\mathbf{fst} \in R$. This follows from the compatibility of R with (A_0, A_1) and the fact that $e_0.\mathbf{fst} \sim e_1.\mathbf{fst} \in \mathcal{Ne}$.
      ii. We check $n \Vdash \underline{\mathbf{snd}}(\uparrow^{\mathbf{\Sigma}(A_0,B_0)} e_0) \sim \underline{\mathbf{snd}}(\uparrow^{\mathbf{\Sigma}(A_0,B_0)} e_1) \in S(\uparrow^{A_0} e_0.\mathbf{fst}, \uparrow^{A_1} e_1.\mathbf{fst})$, which is the same as to say:

$$n \Vdash \uparrow^{B_0[\uparrow^{A_0} e_0.\mathbf{fst}]} e_0.\mathbf{snd} \sim \uparrow^{B_1[\uparrow^{A_1} e_1.\mathbf{fst}]} e_1.\mathbf{snd} \in S(\uparrow^{A_0} e_0.\mathbf{fst}, \uparrow^{A_1} e_1.\mathbf{fst})$$

      Observing that $e_0.\mathbf{snd} \sim e_1.\mathbf{snd} \in \mathcal{Ne}$, it would suffice to show that the fiber $S(\uparrow^{A_0} e_0.\mathbf{fst}, \uparrow^{A_1} e_1.\mathbf{fst})$ is compatible with $(B_0[\uparrow^{A_0} e_0.\mathbf{fst}], B_1[\uparrow^{A_1} e_1.\mathbf{fst}])$. This would follow from our induction hypothesis, if we could show that $n \Vdash \uparrow^{A_0} e_0.\mathbf{fst} \sim \uparrow^{A_1} e_1.\mathbf{fst} \in R$; this follows from the compatibility of R with (A_0, A_1) and the fact that $e_0.\mathbf{fst} \sim e_1.\mathbf{fst} \in \mathcal{Ne}$.

b) Suppose that $n \Vdash u_0 \sim u_1 \in [\![\Sigma]\!](R,S)$; we need to show that $\downarrow^{\Sigma(A_0,B_0)} u_0 \sim \downarrow^{\Sigma(A_1,B_1)} u_1 \in \mathcal{N}\!f$. This reduces to two subproblems:

i. First, we need to show that $\downarrow^{A_0} \underline{\mathbf{fst}}(u_0) \sim \downarrow^{A_1} \underline{\mathbf{fst}}(u_1) \in \mathcal{N}\!f$. By assumption, we have $n \Vdash \underline{\mathbf{fst}}(u_0) \sim \underline{\mathbf{fst}}(u_1) \in R$, and so our goal follows from the compatibility of R with (A_0, A_1).

ii. Second, we need to show that $\downarrow^{B_0[\underline{\mathbf{fst}}(u_0)]} \underline{\mathbf{snd}}(u_0) \sim \downarrow^{B_1[\underline{\mathbf{fst}}(u_1)]} \underline{\mathbf{snd}}(u_1) \in \mathcal{N}\!f$. First, observe that $S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_1))$ is compatible with $(B_0[\underline{\mathbf{fst}}(u_0)], B_1[\underline{\mathbf{fst}}(u_1)])$, following from the fact that $n \Vdash \underline{\mathbf{fst}}(u_0) \sim \underline{\mathbf{fst}}(u_1) \in R$. Therefore, our goal follows from our assumption that $n \Vdash \underline{\mathbf{snd}}(u_0) \sim \underline{\mathbf{snd}}(u_1) \in S(\underline{\mathbf{fst}}(u_0), \underline{\mathbf{fst}}(u_1))$.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad n \Vdash v_0 \sim u_0 \in R \qquad n \Vdash v_1 \sim u_1 \in R}{\mathbf{Id}[\sigma] \models_n \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)}$$

1. We observe that $\mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \in \mathcal{T}\!y$ follows from $A_0 \sim A_1 \in \mathcal{T}\!y$, $\downarrow^{A_0} u_i \sim \downarrow^{A_0} u_i \in \mathcal{N}\!f$, which are all obtained from the induction hypothesis.
2. $\tau_\alpha \models_n \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1)$ follows from the induction hypothesis.
3. Finally, we check that $[\![\mathsf{Id}]\!](R, u_0, u_1)$ is compatible with $(\mathbf{Id}(A_0, v_0, v_1), \mathbf{Id}(A_1, u_0, u_1))$.

a) Suppose that $e_0 \sim e_1 \in \mathcal{N}\!e$; we need to show that $n \Vdash \uparrow^{\mathbf{Id}(A_0,v_0,v_1)} e_0 \sim \uparrow^{\mathbf{Id}(A_1,u_0,u_1)} e_1 \in [\![\mathsf{Id}]\!](R, u_0, u_1)$. This is immediate from the definition of $[\![\mathsf{Id}]\!](R, u_0, u_1)$.

b) Suppose that $n \Vdash v_0 \sim v_1 \in [\![\mathsf{Id}]\!](R, u_0, u_1)$; we need to show that $\downarrow^{\mathbf{Id}(A_0,v_0,v_1)} v_0 \sim \downarrow^{\mathbf{Id}(A_1,u_0,u_1)} v_1 \in \mathcal{N}\!f$. We shall show this by cases on $n \Vdash v_0 \sim v_1 \in [\![\mathsf{Id}]\!](R, u_0, u_1)$.

i. For the first suppose that $u_i = \mathbf{refl}(w_i)$ such that $n \Vdash u_0 \sim w_0 \in R$, $n \Vdash w_0 \sim w_1 \in R$, and $n \Vdash w_1 \sim u_1 \in R$. We wish to show that $\downarrow^{\mathbf{Id}(A_0,v_0,v_1)} v_0 \sim \downarrow^{\mathbf{Id}(A_1,u_0,u_1)} v_1 \in \mathcal{N}\!f$ holds. By inspection on the definition of quotation we see that it is sufficient to show that $\downarrow^{A_0} w_0 \sim \downarrow^{A_1} w_1 \in \mathcal{N}\!f$. This, however, is immediate from our induction hypothesis.

ii. For the second case we suppose that $u_i = \uparrow^{\mathbf{Id}(-,-,-)} e_i$ such that $e_0 \sim e_1 \in \mathcal{N}\!e$. We see by inspection that it suffices to show $e_0 \sim e_1 \in \mathcal{N}\!e$ so this case is immediately satisfied.

Case.

$$\frac{\forall m.\ \sigma \models_m A_0 \sim A_1 \downarrow R(m) \qquad S = \{(n, u_0, u_1) \mid n \Vdash u_0 \sim u_1 \in R(n)\}}{\mathbf{Box}[\sigma] \models_n \Box A_0 \sim \Box A_1 \downarrow [\![\Box]\!](S)}$$

1. We observe that $\Box A_0 \sim \Box A_1 \in \mathcal{T}\!y$ follows from $A_0 \sim A_1 \in \mathcal{T}\!y$, which is obtained from the induction hypothesis.
2. $\tau_\alpha \models_n \Box A_0 \sim \Box A_1$ follows from the induction hypothesis.
3. Finally, we check that $[\![\Box]\!](S)$ is compatible with $(\Box A_0, \Box A_1)$.

a) Suppose that $e_0 \sim e_1 \in \mathcal{N}\!e$; we need to show that $n \Vdash \uparrow^{\Box A_0} e_0 \sim \uparrow^{\Box A_1} e_1 \in [\![\Box]\!](S)$. Unfolding definitions, this means that for all m, we need to show that $m \Vdash \underline{\mathbf{open}}(\uparrow^{\Box A_0} e_0) \sim \underline{\mathbf{open}}(\uparrow^{\Box A_1} e_1) \in R(m)$, which is the same as to say $m \Vdash \uparrow^{A_0} e_0.\mathbf{open} \sim \uparrow^{A_1} e_1.\mathbf{open} \in R(m)$. By the induction hypothesis, we know that each $R(m)$ is compatible with (A_0, A_1), so it suffices to observe that $e_0.\mathbf{open} \sim e_1.\mathbf{open} \in \mathcal{N}\!e$.

b) Suppose that $n \Vdash v_0 \sim v_1 \in [\![\Box]\!](S)$; we need to show that $\downarrow^{\Box A_0} v_0 \sim \downarrow^{\Box A_1} v_1 \in \mathcal{N}\!f$. It suffices to verify that $\downarrow^{A_0} \underline{\mathbf{open}}(v_0) \sim \downarrow^{A_1} \underline{\mathbf{open}}(v_1) \in \mathcal{N}\!f$. Because each $R(n)$ is compatible with (A_0, A_1), we just need to show that $n \Vdash \underline{\mathbf{open}}(v_0) \sim \underline{\mathbf{open}}(v_1) \in R(n)$. But this follows from our assumption that $n \Vdash v_0 \sim v_1 \in [\![\Box]\!](S)$.

Case.

$$\mathbf{Nat} \models_n \mathbf{nat} \sim \mathbf{nat} \downarrow [\![\mathbb{N}]\!]$$

We only need to show that $[\![\mathbb{N}]\!]$ is compatible with $(\mathbf{nat}, \mathbf{nat})$.

1. Suppose that $e_0 \sim e_1 \in \mathcal{N}e$; it is immediate that $n \Vdash \uparrow^{\mathsf{nat}} e_0 \sim \uparrow^{\mathsf{nat}} e_1 \in [\![\mathbb{N}]\!]$ for all n.
2. Suppose that $n \Vdash v_0 \sim v_1 \in \mathbf{nat}$; we need to show that $\downarrow^{\mathsf{nat}} v_0 \sim \downarrow^{\mathsf{nat}} v_1 \in \mathcal{N}f$. We proceed by induction on $n \Vdash v_0 \sim v_1 \in \mathbf{nat}$.
   a) Trivially, we have $\downarrow^{\mathsf{nat}} \mathbf{zero} \sim \downarrow^{\mathsf{nat}} \mathbf{zero} \in \mathcal{N}f$.
   b) Assuming $\downarrow^{\mathsf{nat}} u_0 \sim \downarrow^{\mathsf{nat}} u_1 \in \mathcal{N}f$, we observe that $\downarrow^{\mathsf{nat}} \mathbf{succ}(u_0) \sim \downarrow^{\mathsf{nat}} \mathbf{succ}(u_1) \in \mathcal{N}f$.
   c) Finally, assuming $e_0 \sim e_1 \in \mathcal{N}e$ we verify that $\downarrow^{\mathsf{nat}} \uparrow^{\mathsf{nat}} e_0 \sim \downarrow^{\mathsf{nat}} \uparrow^{\mathsf{nat}} e_1 \in \mathcal{N}f$.

□

Lemma 3.2.8. *$\tau_{(-)}$ is cumulative.*

Proof. In order to show this, first recall that $\mu : (L \longrightarrow L) \longrightarrow L$, the least fixed-point operator, is a monotone function. In order to show that if $i \leq \alpha$ then $\tau_i \leq \tau_\alpha$, therefore, it suffices to show that $\mathbf{Types}_i[\sigma] \leq \mathbf{Types}_\alpha[\sigma]$ for all σ. Examination of the definition of $\mathbf{Types}_i$ and $\mathbf{Types}_\alpha$ shows us that we merely need to show $\mathbf{Univ}_i \leq \mathbf{Univ}_\alpha$ as the rest of the definition is identical.

Suppose that $\mathbf{Univ}_i \models_n A \sim B \downarrow R$, we wish to show $\mathbf{Univ}_\alpha \models_n A \sim B \downarrow R$. Inversion on our premise tells us that we must have some $j < i$ such that $\mathbf{U}_j = A = B$. We must also have that $m \Vdash v_0 \sim v_1 \in R$ if and only if $\tau_j \models_m v_0 \sim v_0$. Since $i \leq \alpha$ we then have that $j \leq \alpha$ and so $\mathbf{Univ}_\alpha \models_n A \sim B \downarrow R$ holds as required. □

3.3 Completeness

In order to prove the fundamental theorem for this logical relation, we must first define a notion of closing substitution. This is somewhat subtle because of the richer notion of context, the indexing, and the dependency.

$$\frac{}{n \Vdash \cdot = \cdot : \cdot} \qquad \frac{n \Vdash \rho_0 = \rho_1 : \Gamma \qquad n \Vdash v_0 = v_1 : A\,[\rho_0; \rho_1]}{n \Vdash \rho_0.v_0 = \rho_1.v_1 : \Gamma.A} \qquad \frac{\exists m.\ m \Vdash \rho_0 = \rho_1 : \Gamma}{n \Vdash \rho_0 = \rho_1 : \Gamma.\text{🔒}}$$

$$\frac{[\![A]\!]_{\rho_0} = A_0 \qquad [\![A]\!]_{\rho_1} = A_1 \qquad \tau_\omega \models_n A_0 \sim A_1 \downarrow R \qquad n \Vdash v_0 \sim v_1 \in R}{n \Vdash v_0 = v_1 : A\,[\rho_1; \rho_2]}$$

Lemma 3.3.1. *For all n and Γ, $n \Vdash - = - : \Gamma$ is a PER on environments.*

Proof. Immediate by induction on Γ with Lemma 3.2.5. □

Lemma 3.3.2. *For Γ, $- \Vdash - = - : \Gamma$ is monotone.*

Proof. Immediate by induction on Γ with Lemma 3.2.5. □

Lemma 3.3.3. *If $n \Vdash \rho_0 = \rho_1 : \Gamma$ then there is some $m \leq n$ such that $m \Vdash \rho_0 = \rho_1 : \Gamma^{\text{🔓}}$.*

Proof. This follows by induction on Γ using Lemma 3.3.2. □

Lemma 3.3.4. *If $\Gamma_0 \triangleright_{\text{🔒}} \Gamma_1$ and $n \Vdash \rho_0 = \rho_1 : \Gamma_0$ then $n \Vdash \rho_0 = \rho_1 : \Gamma_1$.*

Proof. This follows by induction on $\Gamma_0 \triangleright_{\text{🔒}} \Gamma_1$. We show the non-congruence cases.

Case.

$$\frac{}{\Gamma \rhd_{\text{🔒}} \Gamma.\text{🔒}}$$

In this case, suppose we have $n \Vdash \rho_0 = \rho_1 : \Gamma$. We wish to show $n \Vdash \rho_0 = \rho_1 : \Gamma.\text{🔒}$. It suffices to find an m such that $m \Vdash \rho_0 = \rho_1 : \Gamma$ but picking $m = n$ gives this immediately.

Case.

$$\frac{}{\Gamma.\text{🔒}.\text{🔒} \rhd_{\text{🔒}} \Gamma.\text{🔒}}$$

In this case, suppose we have $n \Vdash \rho_0 = \rho_1 : \Gamma.\text{🔒}.\text{🔒}$. We wish to show $n \Vdash \rho_0 = \rho_1 : \Gamma.\text{🔒}$. It suffices to find an m such that $m \Vdash \rho_0 = \rho_1 : \Gamma$. This is immediate, however, by inverting upon $n \Vdash \rho_0 = \rho_1 : \Gamma.\text{🔒}.\text{🔒}$.

Case.

$$\frac{}{\Gamma.\text{🔒}.T \rhd_{\text{🔒}} \Gamma.T.\text{🔒}}$$

In this case, suppose we have $n \Vdash \rho_0.v_0 = \rho_1.v_1 : \Gamma.\text{🔒}.T$. We wish to show $n \Vdash \rho_0 = \rho_1 : \Gamma.T.\text{🔒}$. It suffices to find an o such that $o \Vdash \rho_0.v_0 = \rho_1.v_1 : \Gamma.T$. By inversion on $n \Vdash \rho_0 = \rho_1 : \Gamma.\text{🔒}.T$ we know that there is some m such that $m \Vdash \rho_0 = \rho_1 : \Gamma$ and that $n \Vdash v_0 = v_1 : T\ [\rho_1; \rho_2]$. By Lemma 3.3.2 we then have that $\min(m,n) \Vdash \rho_0.v_0 = \rho_1.v_0 : \Gamma.T$. Choosing $o = \min(m,n)$ gives the desired conclusion. □

Theorem 3.3.5 (Completeness). *The following 6 statements hold.*

1. *If $\Gamma \vdash A$ type and $n \Vdash \rho_0 = \rho_1 : \Gamma$ then there exists A_0, A_1 such that $[\![A]\!]_{\rho_0} = A_0$ and $[\![A]\!]_{\rho_1} = A_1$ and $\tau_\omega \models_n A_0 \sim A_1$.*

2. *If $\Gamma \vdash t : A$ and $n \Vdash \rho_0 = \rho_1 : \Gamma$ then there exists A_0, A_1 and v_0, v_1 such that $[\![A]\!]_{\rho_i} = A_i$, $[\![t]\!]_{\rho_i} = v_i$, and there is an R such that $\tau_\omega \models_n A_0 \sim A_1 \downarrow R$ and $n \Vdash v_0 \sim v_1 \in R$.*

3. *If $\Gamma \vdash \delta : \Delta$ and $n \Vdash \rho_0 = \rho_1 : \Gamma$ then there exists ρ_0', ρ_1' such that $[\![\delta]\!]_{\rho_i} = \rho_i'$ and $n \Vdash \rho_0' = \rho_1' : \Delta$*

4. *If $\Gamma \vdash A_0 = A_1$ type and $n \Vdash \rho_0 = \rho_1 : \Gamma$ then there exists A_0, A_1 such that $[\![A_i]\!]_{\rho_i} = A_i$ and $\tau_\omega \models_n A_0 \sim A_1$.*

5. *If $\Gamma \vdash t_0 = t_1 : A$ and $n \Vdash \rho_0 = \rho_1 : \Gamma$ then there exists A_0, A_1 and v_0, v_1 such that $[\![A]\!]_{\rho_i} = A_i$, $[\![t_i]\!]_{\rho_i} = v_i$, and there is an R such that $\tau_\omega \models_n A_0 \sim A_1 \downarrow R$ and $n \Vdash v_0 \sim v_1 \in R$.*

6. *If $\Gamma \vdash \delta_0 = \delta_1 : \Delta$ and $n \Vdash \rho_0 = \rho_1 : \Gamma$ then there exists ρ_0', ρ_1' such that $[\![\delta_i]\!]_{\rho_i} = \rho_i'$ and $n \Vdash \rho_0' = \rho_1' : \Delta$*

Proof. Completeness is obtained by mutual induction on the derivations; we illustrate the cases of substance. Since all the unary cases are identical to the congruence cases we have elided these.

Case.

$$\frac{\Gamma.\text{🔒} \vdash A_0 = A_1\ \mathit{type}}{\Gamma \vdash \Box A_0 = \Box A_1\ \mathit{type}}$$

Suppose that $n \Vdash \rho_0 = \rho_1 : \Gamma$; we need to show that for some C_i we have $[\![\Box A_i]\!]_{\rho_i} = C_i$ and some R such that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$.

By our induction hypothesis, for all stages m, we have some A^i_m, S_m such that $[\![A_i]\!]_{\rho_i} = A^i_m$ and $\tau_\omega \models_m A^0_m \sim A^i_m \downarrow S_m$. By the determinacy of evaluation, we can that A^i_m do not vary in m, so we are justified in calling them A_i. Using the determinacy of the type system and the constructive axiom of unique choice, we furthermore obtain in fact a family of relations $S \in \mathbf{Rel}^{\mathbb{P}}$ from the individual relations S_m.

Inspecting the definition of the evaluation relation, we are free to choose $C_i = \Box A_i$, choosing a suitable R as follows:

$$R = [\![\Box]\!](\{(m, v_0, v_1) \mid m \Vdash v_0 \sim v_1 \in S(m)\})$$

It remains to show that $\tau_\omega \models_n \Box A_0 \sim \Box A_1 \downarrow R$; using the closure of the type system under the **Box** operator, we just need to see that $\tau_\omega \models_{m'} A_0 \sim A_1 \downarrow S(m')$ for all stages m'. But this is already contained in the induction hypothesis.

Case.

$$\frac{\Delta.🔒 \vdash A \ type \qquad \Gamma \vdash \delta : \Delta}{\Gamma \vdash (\Box A)[\delta] = \Box A[\delta] \ type}$$

Suppose that $n \Vdash \rho_0 = \rho_1 : \Gamma$; we need to show that for some B_i we have $[\![(\Box A)[\delta]]\!]_{\rho_i} = B_i$ and some R such that $\tau_\omega \models_n B_0 \sim B_1 \downarrow R$.

By our induction hypothesis, we have that there are some σ_i such that $[\![\delta]\!]_{\rho_i} = \sigma_i$ and $n \Vdash \sigma_0 = \sigma_1 : \Delta$. We may use these new environments to instantiate our other induction hypothesis. This tells us that for all stages m we have some A^i_m such that $[\![A]\!]_{\sigma_i} = A^i_m$ and $\tau_\omega \models_m A^0_m \sim B^1_m \downarrow S_m$ for some S_m. By determinacy of evaluation we know that all A^i_ms do not vary in m, so we will henceforth write them as A_i. Likewise, by determinacy we obtain a relation $S \in \mathbf{Rel}^{\mathbb{P}}$ from S_m.

We observe that by calculation $[\![(\Box A)[\delta]]\!]_{\rho_i} = \Box A_i$, leading us to chose $B_i = \Box A_i$. Finally, observe that because τ_ω is closed under **Box** we have $\tau_\omega \models_n \Box A_0 \sim \Box A_1 \downarrow T$ where we have defined T as follows:

$$T = [\![\Box]\!](\{(\_, v_0, v_1) \mid \forall m.\ m \Vdash v_0 \sim v_1 \in S(m)\})$$

Case.

$$\textsc{pi}\ \frac{\Gamma \vdash A_0 = A_1 \ type \qquad \Gamma.A_0 \vdash B_0 = B_1 \ type}{\Gamma \vdash \Pi(A_0, B_0) = \Pi(A_1, B_1) \ type}$$

Fix $n \Vdash \rho_0 = \rho_1 : \Gamma$. We need to show that $[\![\Pi(A_i, B_i)]\!]_{\rho_i} = F_i$ for some F_i such that $\tau_\omega \models_n F_0 \sim F_1$. Unpacking our first induction hypothesis, we have $[\![A_i]\!]_{\rho_i} = A_i$ such that $\tau_\omega \models_n A_0 \sim A_1 \downarrow R$ for some R. We choose $F_i = \mathbf{\Pi}(A_i, B_i \triangleleft \rho_i)$; to verify $\tau_\omega \models_n F_0 \sim F_1$, we will show that $\mathbf{Pi}[\tau_\omega] \models_n F_0 \sim F_1$.

1. We have already seen that $\tau_\omega \models_n A_0 \sim A_1 \downarrow R$.
2. To exhibit $\tau_\omega \models_n R \gg B_0 \triangleleft \rho_0 \sim B_1 \triangleleft \rho_1$, we fix $m \leq n$ and $m \Vdash a_0 \sim a_1 \in R$, to verify that $\tau_\omega \models_m B_0 \triangleleft \rho_0[a_0] \sim B_1 \triangleleft \rho_1[a_1] \downarrow S(a_0, a_1)$ for some $S \in \mathbf{Fam}$.
    a) First, we observe that $m \Vdash \rho_0.a_0 = \rho_1.a_1 : \Gamma.A$ from $m \Vdash a_0 = a_1 : A\ [\rho_0; \rho_1]$, which follows from $m \Vdash a_0 \sim a_1 \in R$, $\tau_\omega \models_m A_0 \sim A_1 \downarrow R$ (by Lemma 3.2.5), and $m \Vdash \rho_0 = \rho_1 : \Gamma$ (by Lemma 3.3.2).
    b) Therefore, by instantiating our second induction hypothesis, there exists some $S_{(a_0,a_1)}$ such that $\tau_\omega \models_m [\![B_0]\!]_{\rho_0.a_0} \sim [\![B_1]\!]_{\rho_1.a1} \downarrow S_{(a_0,a_1)}$, which is the same as $\tau_\omega \models_m B_0 \triangleleft \rho_0[a_0] \sim B_1 \triangleleft \rho_1[a_1] \downarrow S_{(a_0,a_1)}$. By the determinacy of the type system, this actually defines a family $S \in \mathbf{Fam}$.

Case.

$$\frac{\Gamma \vdash A_0 = A_1 : \mathsf{U}_j}{\Gamma \vdash A_0 = A_1 \ type}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![A_i]\!]_{\rho_i} = A_i$ and $\tau_\omega \models_n A_0 \sim A_1$ for some A_i. By the induction hypothesis, we already have $[\![A_i]\!]_{\rho_i} = A_i$ and $[\![\mathsf{U}_j]\!]_{\rho_i} = U_i$ with $\tau_\omega \models_n U_0 \sim U_1 \downarrow S$ for some S and moreover $n \Vdash A_0 \sim A_1 \in S$. By inversion, we observe that $U_i = \mathbf{U}_j$ and $S = (\tau_i \models_{(-)} - \sim -)$. Therefore, we have $\tau_i \models_n A_0 \sim A_1$, and we obtain $\tau_\omega \models_n A_0 \sim A_1$ from Lemma 3.2.8.

Case.

$$\frac{\Gamma = \Gamma_1.A.\Gamma_2 \qquad \|\Gamma_2\| = m \qquad \text{🔒} \notin \Gamma_2}{\Gamma \vdash \mathsf{var}_m = \mathsf{var}_m : A[\mathsf{p}^m]}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![A[\mathsf{p}^m]]\!]_{\rho_i} = A_i$ for some A_i such that $\tau_\omega \models_n A_0 \sim A_1 \downarrow R$ for some R, and moreover, that $[\![\mathsf{var}_m]\!]_{\rho_i} = v_i$ for some v_i such that $n \Vdash v_0 \sim v_1 \in R$. Setting $v_i = \rho_i(m)$, we invert our assumption $n \Vdash \rho_0 = \rho_1 : \Gamma$ to obtain $n \Vdash \rho_0'.v_0 = \rho_1'.v_1 : \Gamma_1.A$ for some ρ_i', whence again by inversion, we have A_i and R with the desired property (using Lemma 2.2.1).

Case.

$$\frac{\Gamma.\text{🔒} \vdash t_0 = t_1 : A}{\Gamma \vdash [t_0]_{\text{🔒}} = [t_1]_{\text{🔒}} : \Box A}\ \textsc{shut}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![\Box A]\!]_{\rho_i} = C_i$ and $[\![[t_i]_{\text{🔒}}]\!]_{\rho_i} = v_i$ such that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$ and $n \Vdash v_0 \sim v_1 \in R$ for some R.

Observing that we have $m \Vdash \rho_0 = \rho_1 : \Gamma.\text{🔒}$ for all m, we obtain by determinacy a family $S \in \mathbf{Rel}^{\mathbb{P}}$ and values A_i, w_i such that $\tau_\omega \models_m A_0 \sim A_1 \downarrow S(m)$ and $[\![A]\!]_{\rho_i} = A_i$ and $[\![t_i]\!]_{\rho_i} = w_i$ and $m \Vdash w_0 \sim w_1 \in S(m)$.

Moreover, by the definition of the evaluation relation, we are constrained to choose $C_i = \Box A_i$ and $v_i = \mathbf{shut}(w_i)$. By the closure of the type system under **Box**, we see that R is likewise constrained, and it remains only to show that for all m, we have $m \Vdash \underline{\mathbf{open}}(\mathbf{shut}(w_0)) \sim \underline{\mathbf{open}}(\mathbf{shut}(w_1)) \in S(m)$. But $\underline{\mathbf{open}}(\mathbf{shut}(w_i)) = w_i$, so we are already done.

Case.

$$\frac{\Gamma^{\text{🔓}} \vdash t_0 = t_1 : \Box A}{\Gamma \vdash [t_0]_{\text{🔓}} = [t_1]_{\text{🔓}} : A}\ \textsc{open}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![A]\!]_{\rho_i} = A_i$ and $[\![[t_i]_{\text{🔓}}]\!]_{\rho_i} = v_i$ for some A_i, v_i such that $\tau_\omega \models_n A_0 \sim A_1 \downarrow R$ $n \Vdash v_0 \sim v_1 \in R$ for some R.

Observe that must exist some m such that $m \Vdash \rho_0 = \rho_1 : \Gamma^{\text{🔓}}$. Then, by the induction hypothesis we have $[\![\Box A]\!]_{\rho_i} = \Box A_i$ and some R such that $\tau_\omega \models_m \Box A_0 \sim \Box A_1 \downarrow R$. Moreover, $[\![t_i]\!]_{\rho_i} = v_i$ for some v_i such that $m \Vdash v_0 \sim v_1 \in R$.

Now, by inversion we must have that $\mathbf{Box}[\tau_\omega] \models_m \Box A_0 \sim \Box A_1 \downarrow R$ and therefore $\mathbf{Box}[\tau_\omega] \models_n \Box A_0 \sim \Box A_1 \downarrow R$. This tells us that there is some $S(m')$ such that $\tau_\omega \models_{m'} A_0 \sim A_1 \downarrow S(m')$ for every m' and, moreover, that $m' \Vdash \underline{\mathbf{open}}(v_0) \sim \underline{\mathbf{open}}(v_1) \in S(m')$. Further inversion tells us that $[\![A]\!]_{\rho_i} = A_i$. Therefore, setting $m' = n$, we obtain the desired conclusion.

Case.

$$\frac{\Gamma^{\text{🔓}}.\text{🔒} \vdash t : A}{\Gamma \vdash [[t]_{\text{🔒}}]_{\text{🔓}} = t : A}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![A]\!]_{\rho_i} = A_i$ and $[\![t]\!]_{\rho_1} = v_1$ and $[\![[[t]_{\text{🔒}}]_{\text{🔓}}]\!]_{\rho_0} = v_0$ for some A_i, v_i such that $\tau_\omega \models_n A_0 \sim A_1 \downarrow R$ and $n \Vdash v_0 \sim v_1 \in R$ for some R.

First, we observe that $n \Vdash \rho_0 = \rho_1 : \Gamma^{\text{🔓}}.\text{🔒}$ using Lemma 3.3.3. Therefore, we may use our induction hypothesis to conclude that $[\![A]\!]_{\rho_i} = A_i$ and $[\![t]\!]_{\rho_i} = v_i$ and for some A_i, v_i such that $\tau_\omega \models_n A_0 \sim A_1 \downarrow R$ and $n \Vdash v_0 \sim v_1 \in R$. Finally , we observe by calculation $[\![[[t]_{\text{🔒}}]_{\text{🔓}}]\!]_{\rho_1} = v_1$.

Case.

$$\frac{\Gamma \vdash t : \Box A}{\Gamma \vdash [[t]_{\text{🔓}}]_{\text{🔒}} = t : \Box A}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![\Box A]\!]_{\rho_i} = C_i$ and $[\![t]\!]_{\rho_1} = v_1$ and $[\![[[t]_{\text{🔓}}]_{\text{🔒}}]\!]_{\rho_0} = v_0$ for some C_i, v_i such that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$ and $n \Vdash v_0 \sim v_1 \in R$ some R.

We use our induction hypothesis to conclude that $[\![\Box A]\!]_{\rho_i} = C_i$ and $[\![t]\!]_{\rho_i} = w_i$ and for some C_i, w_i such that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$ and $n \Vdash w_0 \sim w_1 \in R$. We therefore set $v_1 = w_1$, but still need to obtain an appropriate v_0.

We observe by inversion that $C_i = \Box A_i$ where $[\![A]\!]_{\rho_i} = A_i$. By inversion again, we obtain $\mathbf{Box}[\tau_\omega] \models_n \Box A_0 \sim \Box A_1 \downarrow R$ and $R = [\![\Box]\!](\{(n, u_0, u_1) \mid n \Vdash u_0 \sim u_1 \in S(n)\})$ for some $S \in \mathbf{Rel}^{\mathbb{P}}$ such that $\tau_\omega \models_m A_0 \sim A_1 \downarrow S(m)$ for all m. What remains is the following:

1. We need to see that $[\![[[t]_{\text{🔓}}]_{\text{🔒}}]\!]_{\rho_0} = v_0$ for some v_0. First, we observe that $[\![[t]_{\text{🔓}}]\!]_{\rho_i} = \underline{\mathbf{open}}(w_i)$ and $m \Vdash \underline{\mathbf{open}}(w_0) \sim \underline{\mathbf{open}}(w_1) \in S(m)$ for all m. Therefore, we set $v_0 = \mathbf{shut}(\underline{\mathbf{open}}(w_0))$.
2. Next, we need to see that $n \Vdash v_0 \sim v_1 \in R$; fixing m, this means to show that $m \Vdash \underline{\mathbf{open}}(v_0) \sim \underline{\mathbf{open}}(v_1) \in S(m)$. Calculating, we have $\underline{\mathbf{open}}(v_0) = \underline{\mathbf{open}}(\mathbf{shut}(\underline{\mathbf{open}}(w_0))) = \underline{\mathbf{open}}(w_0)$; but we have already observed that $m \Vdash \underline{\mathbf{open}}(w_0) \sim \underline{\mathbf{open}}(w_1) \in S(m)$.

Case.

$$\frac{\Gamma \vdash A \ \mathit{type} \qquad \Gamma.A \vdash t_0 = t_1 : B}{\Gamma \vdash \lambda(t_0) = \lambda(t_1) : \Pi(A, B)}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![\Pi(A, B)]\!]_{\rho_i} = C_i$ and some C_i such that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$ and $n \Vdash \lambda(t_0 \triangleleft \rho_0) \sim \lambda(t_1 \triangleleft \rho_1) \in R$ for some R. First, we observe that because $\Gamma \vdash A$ *type*, we have $[\![A]\!]_{\rho_i} = A_i$ such that $\tau_\omega \models_n A_0 \sim A_1 \downarrow S$ for some S. Hence we set $C_i = \mathbf{\Pi}(A_i, B \triangleleft \rho_i)$, since $[\![\Pi(A, B)]\!]_{\rho_i} = \mathbf{\Pi}(A_i, B \triangleleft \rho_i)$. What remains is to show the following:

1. $\mathbf{Pi}[\tau_\omega] \models_n \mathbf{\Pi}(A_0, B \triangleleft \rho_0) \sim \mathbf{\Pi}(A_1, B \triangleleft \rho_1) \downarrow R$ for some R. For this, it suffices to show that $\tau_\omega \models_n S \gg B \triangleleft \rho_0 \sim B \triangleleft \rho_1 \downarrow T$ for some family T, but this follows from our second induction hypothesis. We have resolved $R = [\![\Pi]\!](S, T)$.
2. $n \Vdash \lambda(t_0 \triangleleft \rho_0) \sim \lambda(t_1 \triangleleft \rho_1) \in [\![\Pi]\!](S, T)$. Fixing $m \Vdash u_0 \sim u_1 \in S$ for some $m \leq n$, we need to show that $m \Vdash \underline{\mathbf{app}}(\lambda(t_0 \triangleleft \rho_0), u_0) \sim \underline{\mathbf{app}}(\lambda(t_1 \triangleleft \rho_1), u_1) \in S(u_0, u_1)$. Observing that $m \Vdash \rho_0.u_0 = \rho_1.u_1 : \Gamma.A$, we use our second induction hypothesis to conclude that $[\![t_i]\!]_{\rho_i.u_i} = v_i$ for some v_i such that $m \Vdash v_0 \sim v_1 \in S(u_0, u_1)$.

Case.

$$\frac{\Gamma \vdash f_0 = f_1 : \Pi(A, B) \qquad \Gamma \vdash a_0 = a_1 : A}{\Gamma \vdash f_0(a_0) = f_1(a_1) : B[\mathsf{id}.a_0]}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![B[\mathsf{id}.a_0]]\!]_{\rho_i} = C_i$ and $[\![f_i(a_i)]\!]_{\rho_i} = v_i$ for some C_i, v_i such that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$ and $n \Vdash v_0 \sim v_1 \in R$ for some R.

Using our second induction hypothesis, we observe that $[\![A]\!]_{\rho_i} = A_i$ and $[\![a_i]\!]_{\rho_i} = a_i$ for some A_i, a_i, and $\tau_\omega \models_n A_0 \sim A_1 \downarrow S$ with $n \Vdash a_0 \sim a_1 \in S$. Consequently, we further observe that $[\![\Pi(A, B)]\!]_{\rho_i} = \mathbf{\Pi}(A_i, B \triangleleft \rho_i)$, and from our first induction hypothesis, we can conclude that $\tau_\omega \models_n \mathbf{\Pi}(A_0, B \triangleleft \rho_0) \sim \mathbf{\Pi}(A_1, B \triangleleft \rho_1)$. By inversion, we have $\mathbf{Pi}[\tau_\omega] \models_n \mathbf{\Pi}(A_0, B \triangleleft \rho_0) \sim \mathbf{\Pi}(A_1, B \triangleleft \rho_1)$, from which we obtain $\tau_\omega \models_n S \gg B \triangleleft \rho_0 \sim B \triangleleft \rho_1 \downarrow T$ for some family T such that $n \Vdash f_0 \sim f_1 \in [\![\Pi]\!](S, T)$.

By instantiating our type family assumption just obtained above with $n \Vdash a_0 \sim a_1 \in S$, we therefore obtain some D_i such that $B \triangleleft \rho_i[a_i] = D_i$ and $\tau_\omega \models_n D_0 \sim D_1 \downarrow T(a_0, a_1)$. Instantiating with $n \Vdash a_0 \sim a_0 \in S$, we further obtain E_i such that $B \triangleleft \rho_i[a_0] = E_i$ and $\tau_\omega \models_n E_0 \sim E_1 \downarrow T(a_0, a_0)$. Setting $C_0 = D_0$ and $C_1 = E_1$, what remains is the following:

1. To see that $\tau_\omega \models_n D_0 \sim E_1$, we recall that $D_1 = E_0$ and $T(a_0, a_1) = T(a_0, a_0)$. Therefore, we set $R = T(a_0, a_1)$.
2. Because $n \Vdash f_0 \sim f_1 \in [\![\Pi]\!](S, T)$, we obtain $[\![f_i(a_i)]\!]_{\rho_i} = v_i$ where $v_i = \underline{\mathbf{app}}(f_i, a_i)$, such that $n \Vdash v_0 \sim v_1 \in R$.

Case.

$$\frac{\Gamma.A \vdash f : B \qquad \Gamma \vdash a : A}{\Gamma \vdash (\lambda(f))(a) = f[\mathsf{id}.a] : B[\mathsf{id}.a]}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![B[\mathsf{id}.a]]\!]_{\rho_i} = C_i$ and $[\![(\lambda(f))(a)]\!]_{\rho_0} = v_0$ and $[\![f[\mathsf{id}.a]]\!]_{\rho_1} = v_1$ for some C_i, v_i such that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$ and $n \Vdash v_0 \sim v_1 \in R$ for some R. From our induction hypothesis, we obtain $[\![A]\!]_{\rho_i} = A_i$ such that $\tau_\omega \models_n A_0 \sim A_1 \downarrow S$ and $[\![a]\!]_{\rho_i} = a_i$ and $n \Vdash a_0 \sim a_1 \in S$.

Next, we observe that $n \Vdash \rho_0.a_0 = \rho_1.a_1 : \Gamma.A$ by definition; combining this with our second induction hypothesis, we conclude that $[\![B]\!]_{\rho_i.a_i} = B_i$ such that $\tau_\omega \models_n B_0 \sim B_1 \downarrow T$ and $[\![f]\!]_{\rho_i.a_i} = r_i$ and $n \Vdash r_0 \sim r_1 \in T(a_0, a_1)$.

By calculation, we see that $[\![\mathsf{id}.a]\!]_{\rho_i} = \rho_i.a_i$, so we are free to choose $C_i = B_i$ and $R = T(a_0, a_1)$. We merely need to show that $[\![(\lambda(f))(a)]\!]_{\rho_0} = v_0$ and $[\![f[\mathsf{id}.a]]\!]_{\rho_1} = v_1$ for some v_i; but by calculation we have $[\![(\lambda(f))(a)]\!]_{\rho_0} = r_0$ and $[\![f[\mathsf{id}.a]]\!]_{\rho_1} = r_1$.

Case.

$$\frac{\Gamma \vdash f : \Pi(A, B)}{\Gamma \vdash \lambda(f[\mathsf{p}^1](\mathsf{var}_0)) = f : \Pi(A, B)}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![\Pi(A, B)]\!]_{\rho_i} = C_i$ and $[\![\lambda(f[\mathsf{p}^1](\mathsf{var}_0))]\!]_{\rho_0} = v_0$ and $[\![f]\!]_{\rho_1} = v_1$ for some C_i, v_i such that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$ and $n \Vdash v_0 \sim v_1 \in R$ for some R. By inverting our induction hypothesis, we obtain $[\![\Pi(A, B)]\!]_{\rho_i} = \mathbf{\Pi}(A_i, B \triangleleft \rho_i)$ and $[\![A]\!]_{\rho_i} = A_i$ for some A_i such that $\tau_\omega \models_n A_0 \sim A_1 \downarrow S$ and $\tau_\omega \models_n S \gg B \triangleleft \rho_0 \sim B \triangleleft \rho_1 \downarrow T$ for some S, T; and moreover, $[\![f]\!]_{\rho_i} = f_i$ such that $n \Vdash f_0 \sim f_1 \in [\![\Pi]\!](S, T)$. We therefore set $C_i = \mathbf{\Pi}(A_i, B \triangleleft \rho_i)$ and $R = [\![\Pi]\!](S, T)$; we need to show that $n \Vdash \lambda((f[\mathsf{p}^1](\mathsf{var}_0)) \triangleleft \rho_0) \sim f_1 \in [\![\Pi]\!](S, T)$. Fixing $m \le n$ and $m \Vdash a_0 \sim a_1 \in S$, we need to see that $m \Vdash \underline{\mathbf{app}}(\lambda((f[\mathsf{p}^1](\mathsf{var}_0)) \triangleleft \rho_0), a_0) \sim \underline{\mathbf{app}}(f_1, a_1) \in T(a_0, a_1)$. First, we observe that $[\![f[\mathsf{p}^1](\mathsf{var}_0)]\!]_{\rho_0.a_0} = \underline{\mathbf{app}}(f_0, a_0)$ because we already have $[\![f]\!]_{\rho_0} = f_0$; therefore $\underline{\mathbf{app}}(\lambda((f[\mathsf{p}^1](\mathsf{var}_0)) \triangleleft \rho_0), a_0) = \underline{\mathbf{app}}(f_0, a_0)$. So it would suffice to verify that $m \Vdash \underline{\mathbf{app}}(f_0, a_0) \sim \underline{\mathbf{app}}(f_1, a_1) \in T(a_0, a_1)$, which we obtain from the fact that $n \Vdash f_0 \sim f_1 \in [\![\Pi]\!](S, T)$.

Case.

$$\frac{\Gamma \vdash l_0 = l_1 : A \qquad \Gamma.A \vdash B \; type \qquad \Gamma \vdash r_0 = r_1 : B[\mathsf{id}.l_0]}{\Gamma \vdash \langle l_0, r_0 \rangle = \langle l_1, r_1 \rangle : \Sigma(A, B)}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![\Sigma(A, B)]\!]_{\rho_i} = C_i$ and $[\![\langle l_0, r_0 \rangle]\!]_{\rho_0} = v_0$ and $[\![\langle l_0, r_0 \rangle]\!]_{\rho_1} = v_1$ for some C_i, v_i such that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$ and $n \Vdash v_0 \sim v_1 \in R$ for some R.

First, we observe by induction hypothesis from the first premise that there is some R_0 such that $[\![A]\!]_{\rho_i} = A_i$ and $\tau_\omega \models_n A_0 \sim A_1 \downarrow R_0$. Furthermore, our induction hypothesis tells us that $[\![l_i]\!]_{\rho_i} = l_i$ such that $n \Vdash l_1 \sim l_2 \in R_0$.

The induction hypothesis for our seocnd premise to conclude that there is some R_1 such that $\tau_\omega \models_n R_0 \gg B{\lhd}\rho_0 \sim B{\lhd}\rho_1 \downarrow R_1$. Furthermore, we have that $[\![r_i]\!]_{\rho_i} = r_i$ and $n \Vdash r_0 \sim r_1 \in R_1(l_0, l_1)$ from the third induction hypothesis.

We now choose $C_i = \Sigma(A_i, B{\lhd}\rho_i)$ and $R = [\![\Sigma]\!](R_0, R_1)$. The remaining goal, that $n \Vdash \langle l_0, r_0\rangle \sim \langle l_1, r_1\rangle \in R$ is immediate by calculation and our assumptions.

Case.

$$\frac{\Gamma \vdash t : \Sigma(A, B)}{\Gamma \vdash \langle \mathsf{fst}(t), \mathsf{snd}(t)\rangle = t : \Sigma(A, B)}$$

Fixing $n \Vdash \rho_0 = \rho_1 : \Gamma$, we need to show that $[\![\Sigma(A, B)]\!]_{\rho_i} = C_i$ and $[\![t]\!]_{\rho_0} = v_0$ and $[\![\langle \mathsf{fst}(t), \mathsf{snd}(t)\rangle]\!]_{\rho_1} = v_1$ for some C_i, v_i such that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$ and $n \Vdash v_0 \sim v_1 \in R$ for some R.

First, we observe by induction hypothesis from the first premise that there is some R such that $[\![\Sigma(A, B)]\!]_{\rho_i} = D_i$ and $\tau_\omega \models_n D_0 \sim D_1 \downarrow R_0$. By inversion, we see that $[\![\Sigma(A, B)]\!]_{\rho_i} = \Sigma(A_i, B{\lhd}\rho_i)$. Therefore, we have that $R = [\![\Sigma]\!](R_0, R_1)$ for some R_0 such that $\tau_\omega \models_n A_0 \sim A_1 \downarrow R_0$ and $\tau_\omega \models_n B{\lhd}\rho_0 \gg B{\lhd}\rho_1 \sim R_1$. Finally, we must have $[\![t]\!]_{\rho_i} = v_i$ such that $n \Vdash v_0 \sim v_1 \in R$.

We observe by definition that this last fact tells us that $n \Vdash \underline{\mathbf{fst}}(v_0) \sim \underline{\mathbf{fst}}(v_1) \in R_0$ and $n \Vdash \underline{\mathbf{snd}}(v_0) \sim \underline{\mathbf{snd}}(v_1) \in R_1(\underline{\mathbf{fst}}(v_0), \underline{\mathbf{fst}}(v_1))$.

We choose $C_i = D_i$. We have immediately that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$. It suffices to show that there is some w_i such that $[\![t]\!]_{\rho_0} = w_0$ and $[\![\langle \mathsf{fst}(t), \mathsf{snd}(t)\rangle]\!]_{\rho_1} = w_1$ such that $n \Vdash w_0 \sim w_1 \in R$. For this, we set $w_0 = v_0$ and $w_1 = \langle \underline{\mathbf{fst}}(v_0), \underline{\mathbf{snd}}(v_1)\rangle$. The latter is defined by assumption. We have that $n \Vdash w_0 \sim w_1 \in R$ holds by calculation.

Case.

$$\frac{\Gamma \vdash l : A \qquad \Gamma.A \vdash B\ \mathit{type} \qquad \Gamma \vdash r : B[\mathsf{id}.l]}{\Gamma \vdash \mathsf{snd}(\langle l, r\rangle) = r : B[\mathsf{id}.l]}$$

In this case fix $n \Vdash \rho_0 = \rho_1 : \Gamma$. We wish to show that $[\![B[\mathsf{id}.l]]\!]_{\rho_i} = C_i$ such that $\tau_\omega \models_n C_0 \sim C_1 \downarrow R$ for some R. Furthermore, we must show that $[\![\langle l, r\rangle]\!]_{\rho_0} = v_0$ and $[\![r]\!]_{\rho_0} = v_1$ such that $n \Vdash v_0 \sim v_1 \in R$.

First, we observe by induction hypothesis that there is some R_0 such that $[\![A]\!]_{\rho_i} = A_i R_0$ and $[\![l]\!]_{\rho_i} = l_i$ such that $n \Vdash l_0 \sim l_1 \in R_0$. We also have by induction hypothesis that $\tau_\omega \models_n R_0 \gg B{\lhd}\rho_0 \sim B{\lhd}\rho_1 \downarrow R_1$.

We have that $[\![B[\mathsf{id}.l]]\!]_{\rho_i} = D_i$ such that $\tau_\omega \models_n D_0 \sim D_1 \downarrow R_1(l_0, l_1)$. We also have that $[\![r]\!]_{\rho_i} = r_i$ such that $n \Vdash r_0 \sim r_1 \in R_1(l_0, l_1)$. Since we have $\underline{\mathbf{snd}}(\langle l_0, r_0\rangle) = r_0$ we have the desired conclusion by setting $C_i = D_i$ and $R = R_1(l_0, l_1)$.

Case.

$$\frac{\Gamma \vdash A = B : \mathsf{U}_i}{\Gamma \vdash A = B : \mathsf{U}_{i+1}}$$

This is immediate from Lemma 3.2.8.

Case.

$$\frac{\Gamma^{🔓} \vdash \delta_0 = \delta_1 : \Delta}{\Gamma \vdash \delta_0 = \delta_1 : \Delta.🔒}$$

In this case, fix some $n \Vdash \rho_0 = \rho_1 : \Gamma$. We wish to show that $[\![\delta_i]\!]_{\rho_i} = \rho_i'$ such that $n \Vdash \rho_0' = \rho_1' : \Delta.🔒$.

First, we observe that there is some m such that $m \Vdash \rho_0 = \rho_1 : \Gamma^{🔓}$ using Lemma 3.3.3. We may then use our induction hypothesis to conclude that $[\![\delta_i]\!]_{\rho_i} = \rho_i'$ such that $m \Vdash \rho_0' = \rho_1' : \Delta$. By definition, we then have $n \Vdash \rho_0' = \rho_1' : \Delta.🔒$ as required.

□

Lemma 3.3.6. *If Γ ctx then there is some ρ such that $\uparrow\Gamma = \rho$ then $n \Vdash \rho = \rho : \Gamma$.*

Proof. This is immediate by induction on Γ using Lemma 3.2.7. □

Corollary 3.3.7. *If $\Gamma \vdash t_0 = t_1 : T$ then $\underline{\mathbf{nbe}}_\Gamma^T(t_i) = t'$ for some t'.*

Proof. If $\Gamma \vdash t_0 = t_1 : T$ then there is some ρ such that $\uparrow\Gamma = \rho$ and $n \Vdash \rho = \rho : \Gamma$ by Lemma 3.3.6. We therefore may apply Theorem 3.3.5 to conclude that there is some A such that $[\![T]\!]_\rho = A$ and $\tau_\omega \models_n A \sim A \downarrow R$. We also have that $[\![t_i]\!]_\rho = v_i$ such that $n \Vdash v_0 \sim v_1 \in R$.

Now, by Lemma 3.2.7 we have that R is compatible and so $\downarrow^A v_0 \sim \downarrow^A v_1 \in \mathcal{N}f$. Therefore, there is a particular t' such that $\lceil \downarrow^A v_i \rceil_{\|\Gamma\|} = t'$. By definition, we then have that $\underline{\mathbf{nbe}}_\Gamma^T(t_i) = t'$ as required. □

4 Soundness of Normalization

4.1 A well-ordering on semantic types

In Section 4.2, we will define a logical relation between syntax and semantics, proceeding by induction on the type at which we are comparing things; unfortunately, the induction is not structural, so we need to define an ordering on semantic types such that, for instance, a dependent function type is strictly greater than all instantiations of its codomain.

We define the order $\sigma \models_n A < B$ on semantic types as the least relation closed under the following rules:

$$\frac{\sigma \models_n A \leq B}{\sigma \models_n A < \Box B} \qquad \frac{\sigma \models_n A \leq B}{\sigma \models_n A < \Sigma(B, C)} \qquad \frac{\sigma \models_n A \leq B}{\sigma \models_n A < \Pi(B, C)}$$

$$\frac{\sigma \models_n B \sim B \downarrow R \qquad m \leq n \qquad m \Vdash a \sim a \in R \qquad \sigma \models_m A \leq C[a]}{\sigma \models_n A < \Sigma(B, C)}$$

$$\frac{\sigma \models_n B \sim B \downarrow R \qquad m \leq n \qquad m \Vdash a \sim a \in R \qquad \sigma \models_m A \leq C[a]}{\sigma \models_n A < \Pi(B, C)} \qquad \frac{\sigma \models_n A \leq B}{\sigma \models_n A < \mathbf{Id}(B, v_0, v_1)}$$

Lemma 4.1.1. *If $\tau_\alpha \models_{n+1} A < B$ then $\tau_\alpha \models_n A < B$.*

Proof. By induction. □

Theorem 4.1.2. *If $\tau_\alpha \models_n A \sim A$, then there is no infinite descending chain in $\alpha \models_n - < -$ starting with A.*

Proof. This is done by showing that the following $\sigma \in \mathbf{Sys}$ is a pre-fixed point of $\mathbf{Types}_\alpha$:

$$\frac{\tau_\alpha \models_n A_0 \sim A_1 \downarrow R \qquad \text{there is no infinite chain starting from } A_0 \text{ with } \tau_\alpha \models_n - < -}{\sigma \models_n A_0 \sim A_1 \downarrow R}$$

We show only the non-trivial cases. Suppose that $\mathbf{Types}_\alpha[\sigma] \models_n A_0 \sim A_1 \downarrow R$ holds; we wish to show that $\sigma \models_n A_0 \sim A_1 \downarrow R$.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow S \qquad \sigma \models_n S \gg B_0 \sim B_1 \downarrow T}{\mathbf{Pi}[\sigma] \models_n \Pi(A_0, B_0) \sim \Pi(A_1, B_1) \downarrow [\![\Pi]\!](S, T)}$$

We now wish to show that $\sigma \models_n \Pi(A_0, B_0) \sim \Pi(A_1, B_1) \downarrow [\![\Pi]\!](S, T)$. We note that $\tau_\alpha \models_n \Pi(A_0, B_0) \sim \Pi(A_1, B_1) \downarrow [\![\Pi]\!](S, T)$ by unfolding the definition of σ in our two assumptions. We merely need to show that there is no infinite chain starting from $\Pi(A_0, B_0)$.

Suppose such a chain exists: $(C_i)_{i \in \mathbb{N}}$ with $\tau_\alpha \models_n C_{i+1} < C_i$ and $C_0 = \Pi(A_0, B_0)$. There are two possible first links in such a chain; we proceed by case.

1. $\textbf{Types}_\alpha \models_n C_1 \leq A_0$. In this case, we would then have that there is an infinite descending chain starting with A_0. This contradicts $\sigma \models_n A_0 \sim A_1 \downarrow S$.
2. There is some $m \leq n$ and $m \Vdash v_0 \sim v_1 \in S$ and $\textbf{Types}_\alpha \models_m C_1 \leq B_0[v_0]$. First, we observe that in this case $\sigma \models_m B_0[v_0] \sim B_1[v_1]$. Next, by Lemma 4.1.1 we observe that $(C_i)_{i\in\mathbb{N}}$ is an infinite descending chain for $\tau_\alpha \models_m - < -$ as well. Therefore, if such a chain exists then it is an infinite descending chain for $\tau_\alpha \models_m - < -$ starting with $B_0[v_0]$. However, this contradicts with our assumption that $\sigma \models_m B_0[v_0] \sim B_1[v_1]$.

Case.

$$\frac{\forall m.\ \sigma \models_m A \sim B}{\textbf{Box}[\sigma] \models_n \Box A \sim \Box B \downarrow R}$$

We now wish to show that $\sigma \models_n \Box A \sim \Box B \downarrow R$.

Let us first observe that $\tau_\alpha \models_n \Box A \sim \Box B \downarrow R$ holds as $\sigma \leq \tau_\alpha$.

Next, we wish to show that there is no infinite descending chain starting from $\Box A$. Suppose that such a chain exists: $(C_i)_{i\in\mathbb{N}}$ with $\tau_\alpha \models_n C_{i+1} < C_i$ and $C_0 = \Box A$. We observe that since $\tau_\alpha \models_n C_1 < \Box A$ it must be that $\tau_\alpha \models_n C_1 \leq A$. Therefore, $(C_i)_{i>0}$ and A is an infinite descending chain starting with A. This contradicts $\sigma \models_m A \sim B$. □

Corollary 4.1.3. *The ordering* $\tau_\alpha \models - < - \iff \exists m.\ \tau_\alpha \models_m - < -$ *is well-founded on semantic types at stage 0.*

Proof. This follows from Lemma 4.1.1 as well as the fact that $\mathbb{N}$ is well-founded. □

We note that this well-ordering of semantic types is also used implicitly by Coq's termination checker in Wieczorek and Biernacki [WB18]; we have explained it explicitly in order to make the mathematical content clear in the absence of a formalization.

4.2 *The logical relation for soundness*

In order to prove soundness we use a logical relation. Essentially we tie together a syntactic value with its counterpart in the model and show that a value related to a term quotes to that term. We then prove the "fundamental theorem" which in this case proves that a term is related to its evaluation. This part is complicated by the necessity of including a Kripke world again so that this logical relation is fibered over the product of contexts and n.

We define the relation $\Gamma \vdash_n t : A \circledR v \in_\alpha A$ and $\Gamma \vdash_n A \circledR A\ \mathsf{type}_\alpha$ by mutual induction. The first relation states that a syntactic term is related to a value at some semantic type where the logical relation has been constructed for the first α universes. The second states that a syntactic type is related to a semantic type but again only considering the first α universes. In order to make this definition work, we must ensure that these relations are monotone with respect to α, n, and Γ. On contexts, we define an order $r : \Gamma \leq \Gamma'$ when Γ is a weakening of Γ'. Weakenings, r, are a special case of substitutions where we restrict the extension rule to only allow the adjoining of variables and remove $\cdot$ and p^i. This means that weakenings may extend the identity substitution by variables and are closed under composition.

We will then prove a property akin to compatibility: suppose that $\forall n.\ \Gamma \vdash_n A \circledR A\ \mathsf{type}_\alpha$ then:

1. $(\forall n.\ \Gamma \vdash_n t : A \circledR v \in_\alpha A)$ then $\lceil \downarrow^A v \rceil_{\|\Gamma\|} = t'$ for some t' and $\Gamma \vdash t = t' : A$.
2. If $\lceil e \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : A$ then $\Gamma \vdash_n t : A \circledR \uparrow^A e \in_\alpha A$.

The induction used to define logical relations is complicated so we take a moment now to explicitly state what is going on. We simultaneously define $- \vdash_n - : - Ⓡ - \in_\alpha A$ and $- \vdash_n - Ⓡ A \ \mathsf{type}_\alpha$ for all α, A, and n such that $\tau_\alpha \models_n A \sim A$. The ordering on the triple (α, A, n) is given as follows:

$$\frac{\beta < \alpha}{(\beta, B, m) < (\alpha, A, n)} \qquad \frac{\tau_\alpha \models_{\min(m,n)} B < A}{(\alpha, B, m) < (\alpha, A, n)}$$

This is not quite a lexicographical ordering, because the type systems are constrained to be equal in the second clause. However, it is clearly stricter than the lexicographical ordering of two well-founded orderings and so is itself well-founded. The crucial move here is that (assuming that types are valid at all the appropriate worlds) we can move to a semantically smaller type and mostly ignore the index.

Logical relation on types Presupposing $\tau_\alpha \models_n C \sim C$, we define $\Gamma \vdash_n C Ⓡ C \ \mathsf{type}_\alpha$ to hold just when one of the following cases applies:

- $\Gamma \vdash_n C Ⓡ \mathbf{nat} \ \mathsf{type}_\alpha$ if $\Gamma \vdash C = \mathsf{nat}$ *type*.
- $\Gamma \vdash_n C Ⓡ \mathbf{\Pi}(A, B) \ \mathsf{type}_\alpha$ if:
  - $\Gamma \vdash C = \Pi(A, B)$ *type* for some A, B;
  - $\Gamma \vdash_n A Ⓡ A \ \mathsf{type}_\alpha$;
  - if $n' \leq n$ and $r : \Gamma' \leq \Gamma$, then $\Gamma' \vdash_{n'} t : A[r] Ⓡ a \in_\alpha A$ implies $\Gamma' \vdash_{n'} B[r.t] Ⓡ B[a] \ \mathsf{type}_\alpha$.
- $\Gamma \vdash_n C Ⓡ \mathbf{\Sigma}(A, B) \ \mathsf{type}_\alpha$ if:
  - $\Gamma \vdash C = \Sigma(A, B)$ *type* for some A, B;
  - $\Gamma \vdash_n A Ⓡ A \ \mathsf{type}_\alpha$;
  - if $n' \leq n$ and $r : \Gamma' \leq \Gamma$, then $\Gamma' \vdash_{n'} t : A[r] Ⓡ a \in_\alpha A$ implies $\Gamma' \vdash_{n'} B[r.t] Ⓡ B[a] \ \mathsf{type}_\alpha$.
- $\Gamma \vdash_n C Ⓡ \mathbf{Id}(A, v_0, v_1) \ \mathsf{type}_\alpha$ if:
  - $\Gamma \vdash C = \mathsf{Id}(A, t_0, t_1)$ *type* for some A, t_0, t_1;
  - $\Gamma \vdash_n A Ⓡ A \ \mathsf{type}_\alpha$;
  - $\Gamma \vdash_n t_i : A Ⓡ v_i \in_\alpha A$ for $i \in \{0, 1\}$.
- $\Gamma \vdash_n C Ⓡ \Box A \ \mathsf{type}_\alpha$ if:
  - $\Gamma \vdash C = \Box A$ *type* for some A;
  - for all m, $\Gamma.🔒 \vdash_m A Ⓡ A \ \mathsf{type}_\alpha$.
- $\Gamma \vdash_n C Ⓡ \uparrow^{-} e \ \mathsf{type}_\alpha$ if, when $r : \Gamma' \leq \Gamma$, there exists C' such that $\lceil e \rceil_{\|\Gamma'\|} = C'$ and $\Gamma' \vdash C[r] = C'$ *type*.
- $\Gamma \vdash_n C Ⓡ \mathbf{U}_j \ \mathsf{type}_\alpha$ if $j < \alpha$ and $\Gamma \vdash C = \mathsf{U}_j$ *type*.

Logical relation on terms Presupposing $\tau_\alpha \models_n C \sim C \downarrow R$, we define $\Gamma \vdash_n t : C \circledR v \in_\alpha C$ to hold just when one of the following cases is applicable:

- $\Gamma \vdash_n t : C \circledR v \in_\alpha \mathbf{nat}$ if:
  - $n \Vdash v \sim v \in R$;
  - $\Gamma \vdash C = \mathsf{nat}\ \mathit{type}$;
  - one of the following three cases is applicable:
    1. $v = \mathbf{zero}$ and $\Gamma \vdash t = \mathsf{zero} : C$;
    2. $v = \mathbf{succ}(v')$, $\Gamma \vdash t = \mathsf{succ}(t') : C$, and $\Gamma \vdash_n t' : C \circledR v' \in_\alpha \mathbf{nat}$;
    3. $v = \uparrow^- e$ and if $r : \Gamma' \leq \Gamma$ then $\lceil e \rceil_{\|\Gamma'\|} = t'$ and $\Gamma' \vdash t[r] = t' : \mathsf{nat}$.
- $\Gamma \vdash_n t : C \circledR v \in_\alpha \mathbf{\Pi}(A, B)$ if:
  - $n \Vdash v \sim v \in R$ and $\Gamma \vdash t : C$;
  - $\Gamma \vdash C = \Pi(A, B)\ \mathit{type}$ for some A, B;
  - $\Gamma \vdash_n A \circledR A\ \mathsf{type}_\alpha$;
  - if $n' \leq n$ and $r : \Gamma' \leq \Gamma$ then $\Gamma' \vdash_{n'} t' : A[r] \circledR a \in_\alpha A$ implies $\Gamma' \vdash_{n'} t[r](t') : B[r.t'] \circledR \underline{\mathbf{app}}(v, a) \in_\alpha B[a]$.
- $\Gamma \vdash_n t : C \circledR v \in_\alpha \mathbf{\Sigma}(A, B)$ if:
  - $n \Vdash v \sim v \in R$ and $\Gamma \vdash t : C$;
  - $\Gamma \vdash C = \Sigma(A, B)\ \mathit{type}$ for some A, B;
  - if $n' \leq n$ and $r : \Gamma' \leq \Gamma$, then $\Gamma' \vdash_{n'} t' : A[r] \circledR a \in_\alpha A$ implies $\Gamma' \vdash_{n'} B[r.t'] \circledR B[a]\ \mathsf{type}_\alpha$;
  - $\Gamma \vdash_n \mathsf{fst}(t) : A \circledR \underline{\mathbf{fst}}(v) \in_\alpha A$;
  - $\Gamma \vdash_n \mathsf{snd}(t) : B[\mathsf{id}.(\mathsf{fst}(t))] \circledR \underline{\mathbf{snd}}(v) \in_\alpha B[\underline{\mathbf{fst}}(v)]$.
- $\Gamma \vdash_n t : C \circledR v \in_\alpha \mathbf{Id}(A, v_0, v_1)$ if:
  - $n \Vdash v \sim v \in R$ and $\Gamma \vdash t : C$;
  - $\Gamma \vdash C = \mathsf{Id}(A, t_0, t_1)\ \mathit{type}$ for some A, t_0, t_1;
  - $\Gamma \vdash_n A \circledR A\ \mathsf{type}_\alpha$;
  - $\Gamma \vdash_n t_i : A \circledR v_i \in_\alpha A$ for $i \in \{0, 1\}$;
  - one of the following cases applies:
    * $v = \uparrow^- e$ and when $r : \Gamma' \leq \Gamma$, then $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : C[r]$.
    * $\Gamma \vdash t = \mathsf{refl}(t') : C$ and $v = \mathbf{refl}(v')$ for some t', v' such that $\Gamma \vdash t' = t_i : A$.
- $\Gamma \vdash_n t : C \circledR v \in_\alpha \square A$ if:
  - $n \Vdash v \sim v \in R$ and $\Gamma \vdash t : C$;
  - $\Gamma \vdash C = \square A\ \mathit{type}$ for some A
  - for all m, $\Gamma.🔒 \vdash_m [t]_{🔓} : A \circledR \underline{\mathbf{open}}(v) \in_\alpha A$
- $\Gamma \vdash_n t : C \circledR \uparrow^- e_1 \in_\alpha \uparrow^- e_2$ if, when $r : \Gamma' \leq \Gamma$, then $\lceil e_1 \rceil_{\|\Gamma'\|} = t'$ and $\lceil e_2 \rceil_{\|\Gamma'\|} = C'$ such that $\Gamma' \vdash C[r] = C'\ \mathit{type}$ and $\Gamma' \vdash t[r] = t' : C[r]$.
- $\Gamma \vdash_n t : C \circledR v \in_\alpha \mathbf{U}_i$ if:

- $i < \alpha$;
- $n \Vdash v \sim v \in R$;
- $\Gamma \vdash t : C$ and $\Gamma \vdash C = \mathsf{U}_i$ *type*;
- $\Gamma \vdash_n t ® v$ type_i.

We observe that the above is well-defined using Lemma 4.2.1 below.

Lemma 4.2.1. *If* $\Gamma \vdash_n t : T ® v \in_\alpha A$ *then* $\tau_\alpha \models_n A \sim B \downarrow R$ *and* $n \Vdash v \sim v \in R$.

Proof. This follows from the fact that each clause of $\Gamma \vdash_n t : T ® v \in_\alpha A$ requires $n \Vdash v \sim v \in R$. ◻

4.3 Properties of the logical relation

In this section we prove a number of properties of our logical relation we shall use later in proving soundness (Section 4.4).

Lemma 4.3.1. *If* $m \leq n$ *and* $\tau_\alpha \models_n A \sim A$ *then the following two facts hold.*

1. $\Gamma \vdash_n T ® A$ type_α *implies* $m \vdash_\Gamma T ® A$ type_α
2. $\Gamma \vdash_n t : T ® v \in_\alpha A$ *implies* $m \vdash_\Gamma t : T ® v \in_\alpha A$

Proof. This proof is immediate by inspection. ◻

Lemma 4.3.2. *If* $\tau_\alpha \models_n A \sim A$ *then the following two facts hold.*

1. $r : \Gamma' \leq \Gamma$ *and* $\Gamma \vdash_n T ® A$ type_α *implies* $\Gamma' \vdash_m T[r] ® A$ type_α
2. $r : \Gamma' \leq \Gamma$ *and* $\Gamma \vdash_n t : T ® v \in_\alpha A$ *implies* $\Gamma' \vdash_m t[r] : T[r] ® v \in_\alpha A$

Proof. This proof is immediate by the composition of weakenings. ◻

Lemma 4.3.3. *If* $\tau_\alpha \models_n A \sim A$ *and* $\Gamma \vdash_n T ® A$ type_α *then* $\Gamma \vdash T$ *type.*

Proof. We proceed by induction on (α, A, n) using the ordering used in the definition of the logical relation. Suppose that this property holds for all $(\beta, B, m) < (\alpha, A, n)$; we proceed by case on A. Since we have $\tau_\alpha \models_n A \sim A$ many cases may be immediately eliminated. The remaining cases are described below.

Case.

$$\Pi(A_0, A_1)$$

In this case by inversion $\Gamma \vdash_n T ® \Pi(A_0, A_1)$ type_α we must have that the following holds:

- $\Gamma \vdash T = \Pi(T_0, T_1)$ *type* for some T_0 and T_1
- $\Gamma \vdash_n T_0 ® A_0$ type_α
- if $n' \leq n$ and $r : \Gamma' \leq \Gamma$ then $\Gamma' \vdash_{n'} t : T_0[r] ® a \in_\alpha A_0$ implies $\Gamma' \vdash_{n'} T_1[r.t] ® A_1[a]$ type_α

Therefore, we have that there exists T_0 and T_1 such that $\Gamma \vdash T = \Pi(T_0, T_1)$ *type*. By Theorem 1.2.16 we must have that $\Gamma \vdash T$ *type* as required.

Case.

$$\Sigma(A_0, A_1)$$

This case is identical to the case for $\Pi(A_0, A_1)$.

Case.

$$\mathbf{U}_i$$

In this case by inversion on $\Gamma \vdash_n T Ⓡ \mathbf{U}_i\ \mathrm{type}_\alpha$ we have $\Gamma \vdash T = \mathsf{U}_i$ *type* and so $\Gamma \vdash T$ *type* by Theorem 1.2.16.

Case.

$$\Box A'$$

In this case by inversion $\Gamma \vdash_n T Ⓡ \Box A'\ \mathrm{type}_\alpha$ we must have that there is some T' such that $\Gamma \vdash T = \Box T'$ *type*. Therefore, $\Gamma \vdash T$ *type* by Theorem 1.2.16.

Case.

$$\mathbf{Id}(A', v_0, v_1)$$

Identical to the previous case.

Case.

$$\uparrow^{-} e$$

Identical to the previous case.

Case.

$$\mathbf{nat}$$

Identical to the previous case. □

Lemma 4.3.4. *If* $\tau_\alpha \models_n A \sim A$ *and* $\Gamma \vdash_n t : T Ⓡ v \in_\alpha A$ *then* $\Gamma \vdash t : T$.

Proof. This follows by case on A. Every clause of $\Gamma \vdash_n t : T Ⓡ v \in_\alpha A$ includes $\Gamma \vdash t : T$ or that there exists some t' such that $\Gamma \vdash t = t' : T$ so this is immediate using Theorem 1.2.16. □

Lemma 4.3.5. *If* $\tau_\alpha \models_n A \sim B \downarrow R$ *then the following two facts hold:*

1. $\Gamma \vdash_n T Ⓡ A\ \mathrm{type}_\alpha$ *then* $\Gamma \vdash_n T Ⓡ B\ \mathrm{type}_\alpha$
2. $\Gamma \vdash_n t : T Ⓡ v \in_\alpha A$ *then* $\Gamma \vdash_n t : T Ⓡ v \in_\alpha B$
3. $n \Vdash v_1 \sim v_2 \in R$ *and* $\Gamma \vdash_n t : T Ⓡ v_1 \in_\alpha A$ *then* $\Gamma \vdash_n t : T Ⓡ v_2 \in_\alpha A$.

Proof. We proceed by induction on α and we will show the following to be a pre-fixed point:

$$\frac{\begin{array}{l}\tau_\alpha \models_n A \sim B \downarrow R \qquad \forall T, \Gamma, m \le n.\ (\Gamma \vdash_m T Ⓡ A\ \mathrm{type}_\alpha \iff \Gamma \vdash_m T Ⓡ B\ \mathrm{type}_\alpha) \\ (\forall t, T, \Gamma, v, m \le n.\ (\Gamma \vdash_m t : T Ⓡ v \in_\alpha A \iff \Gamma \vdash_m t : T Ⓡ v \in_\alpha B) \\ \forall m \le n, t, T, m \Vdash v_0 \sim v_1 \in R.\ (\Gamma \vdash_m t : T Ⓡ v_0 \in_\alpha A_0 \iff \Gamma \vdash_m t : T Ⓡ v_1 \in_\alpha A_0))\end{array}}{\sigma \models_n A \sim B \downarrow R}$$

In order to do this, we suppose that $\mathbf{Types}_\alpha[\sigma] \models_n A \sim B \downarrow R$. We wish to show $\sigma \models_n A \sim B \downarrow R$; we proceed by cases.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R_0 \qquad \sigma \models_n R_0 \gg B_0[v_0] \sim B_1[v_1] \downarrow R_1}{\mathbf{Pi}[\sigma] \models_n \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1) \downarrow [\![\Pi]\!](R_0, R_1)}$$

We set $R = [\![\Pi]\!](R_0, R_1)$. We to show $\sigma \models_n \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1) \downarrow R$ wish to For this, we must show 4 things.

1. $\sigma \models_n \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1) \downarrow R$.
   This is immediate as we can construct $\mathbf{Pi}[\tau_\alpha] \models_n \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1) \downarrow R$ from our assumptions.

2. For all T, Γ, and $m \leq n$ we have $\Gamma \vdash_m T \circledR \mathbf{\Pi}(A_0, B_0)$ type$_\alpha$ iff $\Gamma \vdash_m T \circledR \mathbf{\Pi}(A_1, B_1)$ type$_\alpha$.
   We assume $\Gamma \vdash_m T \circledR \mathbf{\Pi}(A_0, B_0)$ type$_\alpha$. We wish to show $\Gamma \vdash_n T \circledR \mathbf{\Pi}(A_1, B_1)$ type$_\alpha$. First, we note that $\Gamma \vdash_m T \circledR \mathbf{\Pi}(A_0, B_0)$ type$_\alpha$ is equivalent:

   - $\Gamma \vdash T = \Pi(T_0, T_1)$ *type* for some T_0 and T_1
   - $\Gamma \vdash_m T_0 \circledR A_0$ type$_\alpha$
   - If $m' \leq m$ and $r : \Gamma' \leq \Gamma$ then $\Gamma' \vdash_{m'} t : T_0[r] \circledR a \in_\alpha A_0$ implies $\Gamma' \vdash_{n'} T_1[r.t] \circledR B_0[a]$ type$_\alpha$

   The definition of $\Gamma \vdash_m T \circledR \mathbf{\Pi}(A_1, B_1)$ type$_\alpha$ is almost identical. First, we note that $\Gamma \vdash T = \Pi(T_0, T_1)$ *type* must hold for some T_0 and T_1 so it suffices to show the second half of $\Gamma \vdash_m T \circledR \mathbf{\Pi}(A_1, B_1)$ type$_\alpha$. We have $\Gamma \vdash_m T_0 \circledR A_1$ type$_\alpha$ immediately from $\sigma \models_n A_0 \sim A_1$ and our assumption of $\Gamma \vdash_m T_0 \circledR A_0$ type$_\alpha$.

   We assume we have that $m' \leq m$ and $r : \Gamma' \leq \Gamma$ and $\Gamma' \vdash_{m'} t : T_0[r] \circledR v \in_\alpha A_1$. Now in this case we note that $\sigma \models_n A_0 \sim A_1 \downarrow R_0$ tells us that we may conclude $\Gamma' \vdash_{m'} t : T_0[r] \circledR v \in_\alpha A_0$. Therefore, we have the following:

   $$\Gamma' \vdash_{m'} T_1[r.t] \circledR B_0[v] \text{ type}_\alpha$$

   We observe that from Lemma 4.2.1 to conclude that $m' \Vdash v \sim v \in R_0$. Therefore, we have $\sigma \models_{m'} B_0[v] \sim B_1[v]$. Now, from this we have $\Gamma' \vdash_{n'} T_1[r.t] \circledR B_1[v]$ type$_\alpha$ as required.

   The proof that $\Gamma \vdash_n T \circledR \mathbf{\Pi}(A_1, B_1)$ type$_\alpha$ implies $\Gamma \vdash_n T \circledR \mathbf{\Pi}(A_0, B_0)$ type$_\alpha$ holds mutatis mutandis.

3. For all T, t, Γ, and $m \leq n$ then $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{\Pi}(A_0, B_0)$ iff $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{\Pi}(A_1, B_1)$.
   Suppose we have some T, t, Γ, and $m \leq n$. We will show only that $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{\Pi}(A_0, B_0)$ implies $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{\Pi}(A_1, B_1)$. First, we observe that $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{\Pi}(A_0, B_0)$ holds if and only if the following conditions hold.

   - $m \Vdash v \sim v \in R$ and $\Gamma \vdash t : T$;
   - $\Gamma \vdash T = \Pi(T_0, T_1)$ *type* for some T_0, T_1;
   - $\Gamma \vdash_m T_0 \circledR A_0$ type$_\alpha$;
   - if $m' \leq m$ and $r : \Gamma' \leq \Gamma$ then $\Gamma' \vdash_{m'} t' : T_0[r] \circledR a \in_\alpha A_0$ implies $\Gamma' \vdash_{m'} t[r](t') : T_1[r.t'] \circledR \underline{\mathbf{app}}(v, a) \in_\alpha B_0[a]$.

   We wish to show $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{\Pi}(A_1, B_1)$ which is defined in a similar way. First, we observe that there must be some T_i such that $\Gamma \vdash T = \Pi(T_0, T_1)$ *type*, $\Gamma \vdash t : T$, $m \Vdash v \sim v \in R$ and $\Gamma \vdash_m T_0 \circledR A_1$ type$_\alpha$ from our assumption. Therefore, we merely need to show the following last item in order to establish our goal. Suppose that $m' \leq m$ and $r : \Gamma' \leq \Gamma$ such that $\Gamma' \vdash_{m'} t' : T_0[r] \circledR a \in_\alpha A_1$. We wish to show $\Gamma' \vdash_{m'} t[r](t') : T_1[r.t'] \circledR \underline{\mathbf{app}}(v, a) \in_\alpha B_1[a]$.

   We may now use $\sigma \models_n A_0 \sim A_1$ to conclude that $\Gamma' \vdash_{m'} t' : T_0[r] \circledR a \in_\alpha A_0$. Therefore, we may conclude the following:

   $$\Gamma' \vdash_{m'} t[r](t') : T_1[r.t] \circledR \underline{\mathbf{app}}(v, a) \in_\alpha B_0[a]$$

   However, from $\Gamma' \vdash_{m'} t' : T_1[r] \circledR a \in_\alpha A_1$ we must have that $m' \Vdash a \sim a \in R_0$ from Lemma 4.2.1 and so $\sigma \models_{m'} B_0[a] \sim B_1[a]$. Finally, we may use this to conclude the goal:

   $$\Gamma' \vdash_{n'} t[r](t') : T_1[r.t] \circledR \underline{\mathbf{app}}(v, a) \in_\alpha B_1[a]$$

4. If $m \Vdash v_0 \sim v_1 \in R$ and $m \leq n$ then $\Gamma \vdash_m t : T$ ® $v_0 \in_\alpha A_0$ if and only if $\Gamma \vdash_m t : T$ ® $v_0 \in_\alpha A_0$.
   We will show on the forward direction. Suppose we have $\Gamma \vdash_m t : T$ ® $v_0 \in_\alpha A_0$. We wish to show $\Gamma \vdash_m t : T$ ® $v_1 \in_\alpha A_0$ holds. First, by inversion on $\Gamma \vdash_m t : T$ ® $v_0 \in_\alpha A$ we observe that there must be some T_0 and T_1 such that $\Gamma \vdash T = \Pi(T_0, T_1)$ *type*, $\Gamma \vdash t : T$, $\Gamma \vdash_m T_0$ ® A_0 type_α and $m' \leq m$ and $r : \Gamma' \leq \Gamma$ such that $\Gamma' \vdash_{m'} t' : T_0$ ® $w \in_\alpha A_0$ we have the following:
   $$\Gamma' \vdash_{m'} t[r](t') : T_1[r.t'] \text{ ® } \underline{\mathbf{app}}(v_1, w) \in_\alpha B_0[w]$$
   Now in order to show our goal it suffices to show that have all $m' \leq m$ and $r : \Gamma' \leq \Gamma$ if $\Gamma' \vdash_{m'} t' : T_0$ ® $w \in_\alpha A_0$ then we have the following:
   $$\Gamma' \vdash_{m'} t[r](t') : T_1[r.t'] \text{ ® } \underline{\mathbf{app}}(v_2, w) \in_\alpha B_0[w]$$
   Now, we must have that $m' \Vdash w \sim w \in R_0$ by Lemma 4.2.1. Therefore, we have $m \Vdash \underline{\mathbf{app}}(v_1, w) \sim \underline{\mathbf{app}}(v_2, w) \in R_1(w, w)$. Furthermore, we have $\sigma \models_{m'} B_0[w] \sim B_1[w] \downarrow R_1(w, w)$. By unfolding the definition of σ then, it is apparent that our goal follows from our assumption of $\Gamma' \vdash_{m'} t[r](t') : T_1[r.t']$ ® $\underline{\mathbf{app}}(v_1, w) \in_\alpha B_0[w]$.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R_0 \qquad \sigma \models_n R_0 \gg B_0[v_0] \sim B_1[v_1] \downarrow R_1}{\mathrm{Sg}[\sigma] \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow [\![\Sigma]\!](R_0, R_1)}$$

We set $R = [\![\Sigma]\!](R_0, R_1)$. We to show $\sigma \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow R$. For this, we must show 4 things.

1. $\tau_\alpha \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow R$.
   This is immediate as we can construct $\mathbf{Pi}[\tau_\alpha] \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow R$ from our assumptions.
2. For all T, Γ, and $m \leq n$ we have $\Gamma \vdash_m T$ ® $\Sigma(A_0, B_0)$ type_α iff $\Gamma \vdash_m T$ ® $\Sigma(A_1, B_1)$ type_α.
   This case is identical to the corresponding case for $\Pi(-, -)$.
3. For all T, t, Γ, and $m \leq n$ then $\Gamma \vdash_m t : T$ ® $v \in_\alpha \Sigma(A_0, B_0)$ iff $\Gamma \vdash_m t : T$ ® $v \in_\alpha \Sigma(A_1, B_1)$.
   Suppose we have some T, t, Γ, and $m \leq n$. We will show only that $\Gamma \vdash_m t : T$ ® $v \in_\alpha \Sigma(A_0, B_0)$ implies $\Gamma \vdash_m t : T$ ® $v \in_\alpha \Sigma(A_1, B_1)$. First, we observe that $\Gamma \vdash_m t : T$ ® $v \in_\alpha \Sigma(A_0, B_0)$ is defined as follows:
   - $m \Vdash v \sim v \in R$ and $\Gamma \vdash t : T$;
   - $\Gamma \vdash T = \Sigma(T_0, T_1)$ *type* for some T_0, T_1;
   - if $m' \leq m$ and $r : \Gamma' \leq \Gamma$, then $\Gamma' \vdash_{m'} t' : T_0[r]$ ® $a \in_\alpha A_0$ implies $\Gamma' \vdash_{m'} T_1[r.t']$ ® $B_0[a]$ type_α;
   - $\Gamma \vdash_m \mathsf{fst}(t) : T_0$ ® $\underline{\mathbf{fst}}(v) \in_\alpha A_0$;
   - $\Gamma \vdash_m \mathsf{snd}(t) : T_1[\mathsf{id}.(\mathsf{fst}(t))]$ ® $\underline{\mathbf{snd}}(v) \in_\alpha B_0[\underline{\mathbf{fst}}(v)]$.

   We wish to show $\Gamma \vdash_m t : T$ ® $v \in_\alpha \Sigma(A_1, B_1)$. First, we observe that T_0 and T_1 such that $\Gamma \vdash T = \Sigma(T_0, T_1)$ *type*, $\Gamma \vdash t : T$, and $m \Vdash v \sim v \in R$. We wish to show that the following three facts hold:
   a) if $m' \leq m$ and $r : \Gamma' \leq \Gamma$, then $\Gamma' \vdash_{m'} t' : T_0[r]$ ® $a \in_\alpha A_1$ implies $\Gamma' \vdash_{m'} T_1[r.t']$ ® $B_1[a]$ type_α;
   b) $\Gamma \vdash_m \mathsf{fst}(t) : T_0$ ® $\underline{\mathbf{fst}}(v) \in_\alpha A_1$;
   c) $\Gamma \vdash_m \mathsf{snd}(t) : T_1[\mathsf{id}.(\mathsf{fst}(t))]$ ® $\underline{\mathbf{snd}}(v) \in_\alpha B_1[\underline{\mathbf{fst}}(v)]$.

The first fact is precisely our induction hypothesis. For the second, we note that since $\sigma \models_m A_0 \sim A_1$ we have the first fact from $\Gamma \vdash_m \mathsf{fst}(t) : T_0 ® \underline{\mathbf{fst}}(v) \in_\alpha A_1$. For the second, we observe that $m \Vdash v \sim v \in R_1$ and so $\sigma \models_m B_0[\underline{\mathbf{fst}}(v)] \sim B_1[\underline{\mathbf{fst}}(v)]$ holds. The second fact follows from this.

4. If $m \leq n$, $m \Vdash v_0 \sim v_1 \in R$ then $\Gamma \vdash_m t : T ® v_0 \in_\alpha \Sigma(A_0, B_0)$ iff $\Gamma \vdash_m t : T ® v_1 \in_\alpha \Sigma(A_0, B_0)$.
   We will show on the forward direction. We wish to show $\Gamma \vdash_m t : T ® v_1 \in_\alpha \Sigma(A_0, B_0)$. Now, inversion on the assumption tells the following:
   - $n \Vdash v_0 \sim v_0 \in R$ and $\Gamma \vdash t : T$;
   - $\Gamma \vdash T = \Sigma(T_0, T_1)$ *type* for some T_0, T_1;
   - if $m' \leq m$ and $r : \Gamma' \leq \Gamma$, then $\Gamma' \vdash_{m'} t' : T_0[r] ® a \in_\alpha A_0$ implies $\Gamma' \vdash_{m'} T_1[r.t'] ® B_0[a]$ type_α;
   - $\Gamma \vdash_m \mathsf{fst}(t) : T_0 ® \underline{\mathbf{fst}}(v_0) \in_\alpha A_0$;
   - $\Gamma \vdash_m \mathsf{snd}(t) : T_1[\mathsf{id}.(\mathsf{fst}(t))] ® \underline{\mathbf{snd}}(v_0) \in_\alpha B_0[\underline{\mathbf{fst}}(v)]$.

   In order to show the goal then it suffices to show the following facts (the rest are identical to our assumptions)
   - $\Gamma \vdash_m \mathsf{fst}(t) : T_0 ® \underline{\mathbf{fst}}(v_1) \in_\alpha A_0$;
   - $\Gamma \vdash_m \mathsf{snd}(t) : T_1[\mathsf{id}.(\mathsf{fst}(t))] ® \underline{\mathbf{snd}}(v_1) \in_\alpha B_0[\underline{\mathbf{fst}}(v)]$.

   First, we observe that $m \Vdash \underline{\mathbf{fst}}(v_0) \sim \underline{\mathbf{fst}}(v_1) \in R_0$ since $n \Vdash v_0 \sim v_1 \in R$ and R is monotone by Lemma 3.2.5.
   Next, since $\sigma \models_m A_0 \sim A_1 \downarrow R_1$ (again using monotonicity) we have the first fact from our assumption that $\Gamma \vdash_m \mathsf{fst}(t) : T_0 ® \underline{\mathbf{fst}}(v_0) \in_\alpha A_0$.
   The second fact is more difficult: we have $m \Vdash \underline{\mathbf{snd}}(v_0) \sim \underline{\mathbf{snd}}(v_1) \in R_1(\underline{\mathbf{fst}}(v_0), \underline{\mathbf{fst}}(v_1))$ and $\sigma \models_m B_0[v_0] \sim B_1[v_1] \downarrow R_1(\underline{\mathbf{fst}}(v_0), \underline{\mathbf{fst}}(v_1))$. Therefore, we may conclude the following:

   $$\Gamma \vdash_m \mathsf{snd}(t) : T_1[\mathsf{id}.(\mathsf{fst}(t))] ® \underline{\mathbf{snd}}(v_1) \in_\alpha B_0[\underline{\mathbf{fst}}(v_0)]$$

   By induction hypothesis it suffices to show $\tau_\alpha \models_m B_0[\underline{\mathbf{fst}}(v_0)] \sim B_0[\underline{\mathbf{fst}}(v_1)]$. However, we know that $\tau_\alpha \models_m B_0[\underline{\mathbf{fst}}(v_1)] \sim B_1[\underline{\mathbf{fst}}(v_1)]$ by assumption and so Lemma 3.2.5 gives the desired conclusion.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad n \Vdash v_0 \sim u_0 \in R \qquad n \Vdash v_1 \sim u_1 \in R}{\mathbf{Id}[\sigma] \models_n \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)}$$

We wish to show $\sigma \models_n \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)$. This requires showing three facts.

1. $\tau_\alpha \models_n \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)$
   In this case we observe that we have $\sigma \models_n A_0 \sim A_1 \downarrow R$, $n \Vdash v_0 \sim u_0 \in R$, and $n \Vdash v_1 \sim u_1 \in R$. From the first fact we have $\tau_\alpha \models_n A_0 \sim A_1 \downarrow R$ and so by closure we have $\tau_\alpha \models_n \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)$.
2. For all T, Γ, and $m \leq n$ we have $\Gamma \vdash_m T ® \mathbf{Id}(A_0, v_0, v_1)$ type_α iff $\Gamma \vdash_m T ® \mathbf{Id}(A_1, u_0, u_1)$ type_α.
   Suppose that we have $m \leq n$ and $\Gamma \vdash_m T ® \mathbf{Id}(A_0, v_0, v_1)$ type_α. By inversion we then have that $\Gamma \vdash T = \mathsf{Id}(T', t_0, t_1)$ *type* such that $\Gamma \vdash_m T' ® A_0$ type_α and $\Gamma \vdash_m t_i : T' ® v_i \in_\alpha A_0$ for $i \in \{0, 1\}$.
   We have $\Gamma \vdash_m T' ® A_1$ type_α as $\sigma \models_n A_0 \sim A_1 \downarrow R$. Next, we use this fact again to conclude that $\Gamma \vdash_m t_i : T' ® u_i \in_\alpha A_1$ for $i \in \{0, 1\}$. Therefore, we have by definition that $\Gamma \vdash_m \mathsf{Id}(T', t_0, t_1) ® \mathbf{Id}(A_1, u_0, u_1)$ type_α

3. For all T, t, Γ, and $m \leq n$ then $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$ iff $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{Id}(A_1, u_0, u_1)$.
   We will show only the forward direction, so suppose that $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$. We wish to show $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{Id}(A_1, u_0, u_1)$. First, we observe by inversion on $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$ to conclude the following:
   - $m \Vdash v \sim v \in [\![\mathsf{Id}]\!](R, u_0, u_1)$ and $\Gamma \vdash t : T$;
   - $\Gamma \vdash T = \mathsf{Id}(T', t_0, t_1)$ *type* for some T', t_0, t_1;
   - $\Gamma \vdash_m T' \circledR A_0$ type_α;
   - $\Gamma \vdash_m t_i : T' \circledR v_i \in_\alpha A_0$ for $i \in \{0, 1\}$;
   - one of the following cases applies:
     - $v = \uparrow^{-} e$ and when $r : \Gamma' \leq \Gamma$, then $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$.
     - $\Gamma \vdash t = \mathsf{refl}(t') : T$ and $v = \mathbf{refl}(v')$ for some t', v' such that $\Gamma \vdash t' = t_i : T'$.

   Now, in order to establish $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{Id}(A_1, u_0, u_1)$ we must show then that $\Gamma \vdash_m t_i : T' \circledR v_i \in_\alpha A_0$ for $i \in \{0, 1\}$ but this holds using our assumption that $\sigma \models_m A_0 \sim A_1$.
4. if $m \leq n$ and $m \Vdash w_0 \sim w_1 \in [\![\mathsf{Id}]\!](R, u_0, u_1)$ then $\Gamma \vdash_m t : T \circledR w_0 \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$ iff $\Gamma \vdash_m t : T \circledR w_1 \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$
   We will show only the forward direction. Suppose that $m \leq n$, $m \Vdash w_0 \sim w_1 \in [\![\mathsf{Id}]\!](R, u_0, u_1)$, and $\Gamma \vdash_m t : T \circledR w_0 \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$. We wish to show $\Gamma \vdash_m t : T \circledR w_1 \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$.
   We proceed by inversion on $\Gamma \vdash_m t : T \circledR w_0 \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$ to conclude the following facts must hold:
   - $m \Vdash w_0 \sim w_0 \in [\![\mathsf{Id}]\!](R, u_0, u_1)$ and $\Gamma \vdash t : T$;
   - $\Gamma \vdash T = \mathsf{Id}(T', t_0, t_1)$ *type* for some T', t_0, t_1;
   - $\Gamma \vdash_m T' \circledR A_0$ type_α;
   - $\Gamma \vdash_m t_i : T' \circledR v_i \in_\alpha A_0$ for $i \in \{0, 1\}$;
   - one of the following cases applies:
     - $w_0 = \uparrow^{-} e$ and when $r : \Gamma' \leq \Gamma$, then $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$.
     - $\Gamma \vdash t = \mathsf{refl}(t') : T$ and $w_0 = \mathbf{refl}(v')$ for some t', v' such that $\Gamma \vdash t' = t_i : T'$.

   In order to obtain the desired conclusion, therefore, we merely must show that one of the following facts is true
   - $v = \uparrow^{-} e$ and when $r : \Gamma' \leq \Gamma$, then $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$.
   - $\Gamma \vdash t = \mathsf{refl}(t') : T$ and $w_1 = \mathbf{refl}(v')$ for some t', v' such that $\Gamma \vdash t' = t_i : T'$.

   However, this follows by case on $m \Vdash w_0 \sim w_1 \in [\![\mathsf{Id}]\!](R, u_0, u_1)$ and our assumptions.

Case.

$$\frac{\forall m.\ \sigma \models_n A_0 \sim A_1 \downarrow S(m) \qquad \_ \Vdash v_0 \sim v_1 \in R \iff \forall n.\ n \Vdash A_0 \sim A_1 \in S(n)}{\mathbf{Box}[\sigma] \models_n \Box A_0 \sim \Box A_1 \downarrow R}$$

We wish to show $\sigma \models_n \Box A_0 \sim \Box A_1 \downarrow R$. This requires us to show three facts.

1. $\tau_\alpha \models_n \Box A_0 \sim \Box A_1 \downarrow R$
   In this case, we observe that for all m we have $\sigma \models_m A_0 \sim A_1 \downarrow S(m)$ so $\tau_\alpha \models_m A_0 \sim A_1 \downarrow S(m)$. Therefore $\tau_\alpha \models_n \Box A_0 \sim \Box A_1 \downarrow R$.
2. For all T, Γ, and $m \leq n$ we have $\Gamma \vdash_m T \circledR \Box A_0$ type_α iff $\Gamma \vdash_m T \circledR \Box A_1$ type_α.
   In this case, we will only show the forwards direction. Suppose $\Gamma \vdash_m T \circledR \Box A_0$ type_α holds. We wish to show $\Gamma \vdash_m T \circledR \Box A_1$ type_α. Recall that $\Gamma \vdash_m T \circledR \Box C$ type_α holds if and only if there is some T' such that $\Gamma \vdash T = \Box T'$ *type* and for all m, $\Gamma.\text{🔒} \vdash_m T' \circledR C$ type_α.

By our assumption, we then have some T' such that $\Gamma \vdash T = \Box T'$ *type*. We merely need to show that for all m, $\Gamma.\text{🔒} \vdash_m T' \circledR A_1$ type$_\alpha$. However, since by assumption we have $\Gamma.\text{🔒} \vdash_m T' \circledR A_0$ type$_\alpha$ this follows from the fact that $\sigma \models_m A_0 \sim A_1$.

3. For all T, t, Γ, and $m \leq n$ then $\Gamma \vdash_m t : T \circledR v \in_\alpha \Box A_0$ iff $\Gamma \vdash_m t : T \circledR v \in_\alpha \Box A_1$.
   For this, we will again show only one direction. Suppose that $\Gamma \vdash_m t : T \circledR v \in_\alpha \Box A_0$. Then we may expand this definition to see that it is equivalent to the following conditions:
   - $\Gamma \vdash T = \Box T'$ *type* for some T'
   - $\Gamma \vdash t : T$ and $m \Vdash v \sim v \in R$
   - for all m, $\Gamma.\text{🔒} \vdash_m [t]_{\text{🔓}} : T' \circledR \underline{\mathbf{open}}(v) \in_\alpha A_0$

   Therefore, we have some T' such that $\Gamma \vdash T = \Box T'$ *type* and $\Gamma \vdash t : T$ and $m \Vdash v \sim v \in R$. We therefore merely need to show for any m' that $\Gamma.\text{🔒} \vdash_{m'} [t]_{\text{🔓}} : T' \circledR \underline{\mathbf{open}}(v) \in_\alpha A_1$. However, since $\sigma \models_{m'} A_0 \sim A_1$ and so this follows from $\Gamma.\text{🔒} \vdash_m [t]_{\text{🔓}} : T' \circledR \underline{\mathbf{open}}(v) \in_\alpha A_0$.
4. for any $m \leq n$ if $m \Vdash v_0 \sim v_1 \in R$ then $\Gamma \vdash_m t : T \circledR v_0 \in_\alpha \Box A_0$ if and only if $\Gamma \vdash_m t : T \circledR v_1 \in_\alpha \Box A_0$.
   For this, we will show only the forward implication. Suppose we have $\Gamma \vdash_m t : T \circledR v_0 \in_\alpha \Box A_0$. By inversion on this fact we have that there is some T' such that $\Gamma \vdash T = \Box T'$ *type* and $\Gamma \vdash t : T$. Furthermore, we have for all m' that $\Gamma.\text{🔒} \vdash_{m'} [t]_{\text{🔓}} : T' \circledR \underline{\mathbf{open}}(v_1) \in_\alpha A_0$.

   We wish to show $\Gamma \vdash_m t : T \circledR v_1 \in_\alpha \Box A_0$. Using the above, it suffices to show for all m' that $\Gamma.\text{🔒} \vdash_{m'} [t]_{\text{🔓}} : T' \circledR \underline{\mathbf{open}}(v_1) \in_\alpha A_0$. However, we have $m' \Vdash \underline{\mathbf{open}}(v_0) \sim \underline{\mathbf{open}}(v_2) \in S(m')$ and $\sigma \models_{m'} A_0 \sim A_1 \downarrow S(m')$. Therefore we have the desired conclusion from the definition of σ.

Case.

$$\frac{e_0 \sim e_1 \in \mathcal{N}e \qquad n \Vdash \uparrow^- e_0' \sim \uparrow^- e_1' \in R \iff e_0' \sim e_1' \in \mathcal{N}e}{\mathbf{Ne}[\sigma] \models_n \uparrow^- e_0 \sim \uparrow^- e_1 \downarrow R}$$

We wish to show $\sigma \models_n \uparrow^- e_0 \sim \uparrow^- e_1 \downarrow R$. In order to do this we first observe that $\tau_\alpha \models_n \uparrow^- e_0 \sim \uparrow^- e_1 \downarrow R$. Furthermore, we have that for any $m \leq n$ that $\Gamma \vdash_m T \circledR \uparrow^- e_0$ type$_\alpha$ is equivalent to the following:

$$\forall r : \Gamma' \leq \Gamma.\ \exists T'.\ \lceil e_1 \rceil_{\|\Gamma'\|} = T' \wedge \Gamma' \vdash T[r] = T'\ \textit{type}$$

However, $e_0 \sim e_1 \in \mathcal{N}e$ and so $\Gamma \vdash_m T \circledR \uparrow^- e_0$ type$_\alpha \iff \Gamma \vdash_m T \circledR \uparrow^- e_1$ type$_\alpha$.

Moreover, $\Gamma \vdash_m t : T \circledR v \in_\alpha \uparrow^- e_0$ if $r : \Gamma' \leq \Gamma$, then $\lceil e_1 \rceil_{\|\Gamma'\|} = t'$ and $\lceil e_2 \rceil_{\|\Gamma'\|} = T'$ such that $\Gamma' \vdash T[r] = T'$ *type* and $\Gamma' \vdash t[r] = t' : T[r]$. However, since $e_1 \sim e_2 \in \mathcal{N}e$ we have that this is precisely equivalent to $\Gamma \vdash_m t : T \circledR v \in_\alpha \uparrow^- e_2$ and we're done.

Finally, if $n \Vdash v_0 \sim v_1 \in R$ then we have that $v_i = \uparrow^- e_i'$ and $e_0 \sim e_1 \in \mathcal{N}e$. Therefore, $\Gamma \vdash_m t : T \circledR \uparrow^- e_0' \in_\alpha \uparrow^- e_0$ if and only if $\Gamma \vdash_m t : T \circledR \uparrow^- e_1' \in_\alpha \uparrow^- e_0$ by calculation.

Case.

$$\frac{i < \alpha}{\mathbf{Univ}_\alpha \models_n \mathbf{U}_i \sim \mathbf{U}_i \downarrow \{(m, A_0, A_1) \mid \tau_j \models_m A_0 \sim A_1\}}$$

Since in this case both sides of the equality are identical all of the conditions are trivial except the last. The last follows by computation.

Case.

$$\frac{}{\mathbf{Nat}[\sigma] \models_n \mathbf{nat} \sim \mathbf{nat} \downarrow [\![\mathbb{N}]\!]}$$

Since in this case both sides of the equality are identical all of the conditions are trivial. □

Lemma 4.3.6. *If* $\tau_\omega \models_{\tau_\alpha} n \sim A \downarrow A$ *and* $\Gamma \vdash T_1 = T_2$ *type then the following two facts hold:*

1. $\Gamma \vdash_n T_1 \circledR A\ \mathsf{type}_\alpha$ *then* $\Gamma \vdash_n T_2 \circledR A\ \mathsf{type}_\alpha$.
2. $\Gamma \vdash_n t : T_1 \circledR v \in_\alpha A$ *then* $\Gamma \vdash_n t : T_2 \circledR v \in_\alpha A$.

Proof. In this case we may observe this by simply case on A (induction is not necessary). In each case the result follows from transitivity of = on types and the conversion rule. □

Lemma 4.3.7. *If* $\tau_\alpha \models_n A \sim A$, $\Gamma \vdash t_1 = t_2 : T$ *and* $\Gamma \vdash_n t_1 : T \circledR v \in_\alpha A$ *then* $\Gamma \vdash_n t_2 : T \circledR v \in_\alpha A$.

Proof. In this case we do need some induction. We proceed by showing that the following is a least pre-fixed point:

$$\frac{\begin{array}{l}\tau_\alpha \models_n A_0 \sim A_1 \downarrow R \\ \forall m \leq n, v, \Gamma, t_1, t_2, T.\ \Gamma \vdash t_1 = t_2 : T \implies (\Gamma \vdash_m t_1 : T \circledR v \in_\alpha A_0 \iff \Gamma \vdash_m t_2 : T \circledR v \in_\alpha A_0)\end{array}}{\sigma \models_n A_0 \sim A_1 \downarrow R}$$

Suppose that we have $\mathbf{Types}_\alpha[\sigma] \models_n A_0 \sim A_1 \downarrow R$. We wish to show $\sigma \models_n A_0 \sim A_1 \downarrow R$.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R_0 \qquad \sigma \models_n R_0 \gg B_0 \sim B_1 \downarrow R_1 \qquad R = [\![\Pi]\!](R_0, R_1)}{\mathbf{Pi}[\sigma] \models_n \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1) \downarrow R}$$

We wish to show $\sigma \models_n \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1) \downarrow R$. This involves showing two facts:

1. $\tau_\alpha \models_n \mathbf{\Pi}(A_0, B_0) \sim \mathbf{\Pi}(A_1, B_1)$
   This is immediate from the fact that $\sigma \leq \tau_\alpha$.
2. for all $m \leq n$, v, Γ, t_1, t_2, T if $\Gamma \vdash t_1 = t_2 : T$ then $\Gamma \vdash_m t_1 : T \circledR v \in_\alpha \mathbf{\Pi}(A_0, B_0)$ iff $\Gamma \vdash_m t_2 : T \circledR v \in_\alpha \mathbf{\Pi}(A_0, B_0)$.
   We will show that $\Gamma \vdash_m t_1 : T \circledR v \in_\alpha \mathbf{\Pi}(A_0, B_0)$ implies $\Gamma \vdash_m t_2 : T \circledR v \in_\alpha \mathbf{\Pi}(A_1, B_1)$. We may unfold $\Gamma \vdash_m t_1 : T \circledR v \in_\alpha \mathbf{\Pi}(A_0, B_0)$ to see that it is equivalent to the following conditions:
   - $n \Vdash v \sim v \in R$ and $\Gamma \vdash t_1 : T$;
   - $\Gamma \vdash T = \Pi(T_0, T_1)$ *type* for some T_0, T_1;
   - $\Gamma \vdash_n T_0 \circledR A_0\ \mathsf{type}_\alpha$;
   - if $m' \leq n$ and $r : \Gamma' \leq \Gamma$ then $\Gamma' \vdash_{m'} t' : T_0[r] \circledR a \in_\alpha A_0$ implies $\Gamma' \vdash_{m'} t[r](t') : T_1[r.t'] \circledR \underline{\mathbf{app}}(v, a) \in_\alpha B_0[a]$.

   The first conditions are identical, therefore, it suffices to show for all $m' \leq m$ and $r : \Gamma' \leq \Gamma$ if $\Gamma' \vdash_{m'} t' : T_1[r] \circledR a \in_\alpha A_1$ then the following:

   $$\Gamma' \vdash_{m'} t_1[r](t') : T_2[r.t'] \circledR \underline{\mathbf{app}}(v, a) \in_\alpha A_2[a]$$

   We must have $m' \Vdash a \sim a \in R$ and so $\sigma \models_{m'} B_0[a] \sim B_1[a] \downarrow R_1(a, a)$. Then, we may conclude from congruence that $\Gamma' \vdash t_1[r](t') = t_2[r](t') : T_1[r.t']$ and so we have the goal:

   $$\Gamma' \vdash_{n'} t_2[r](t') : T_1[r.t'] \circledR \underline{\mathbf{app}}(v, a) \in_\alpha B_1[a]$$

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R_0 \qquad \sigma \models_n R_0 \gg B_0 \sim B_1 \downarrow R_1 \qquad R = [\![\Sigma]\!](R_0, R_1)}{\mathbf{Sg}[\sigma] \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow R}$$

We wish to show $\sigma \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow R$. This involves showing two facts:

1. $\tau_\alpha \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow R$
   This is identical to the reasoning in the **Pi** case.
2. for all $m \leq n$, v, Γ, t_1, t_2, T if $\Gamma \vdash t_1 = t_2 : T$ then $\Gamma \vdash_m t_1 : T \circledR v \in_\alpha \Sigma(A_0, B_0)$ iff $\Gamma \vdash_m t_2 : T \circledR v \in_\alpha \Sigma(A_0, B_0)$.
   We will show that $\Gamma \vdash_m t_1 : T \circledR v \in_\alpha \Sigma(A_0, B_0)$ implies $\Gamma \vdash_m t_2 : T \circledR v \in_\alpha \Sigma(A_0, B_0)$. We may unfold $\Gamma \vdash_m t_1 : T \circledR v \in_\alpha \Sigma(A_0, B_0)$ to see that it is equivalent to the following conditions:
   - $n \Vdash v \sim v \in R$ and $\Gamma \vdash t : T$;
   - $\Gamma \vdash T = \Sigma(T_0, T_1)$ *type* for some T_0, T_1;
   - if $m' \leq m$ and $r : \Gamma' \leq \Gamma$, then $\Gamma' \vdash_{m'} t' : T_0[r] \circledR a \in_\alpha A_0$ implies $\Gamma' \vdash_{m'} T_1[r.t'] \circledR B_0[a]$ type_α;
   - $\Gamma \vdash_n \mathsf{fst}(t) : T_0 \circledR \underline{\mathbf{fst}}(v) \in_\alpha A_0$;
   - $\Gamma \vdash_n \mathsf{snd}(t) : T_1[\mathsf{id}.(\mathsf{fst}(t))] \circledR \underline{\mathbf{snd}}(v) \in_\alpha B_0[\underline{\mathbf{fst}}(v)]$.

   So there exists some T_0, T_1 such that $\Gamma \vdash T = \Sigma(T_0, T_1)$ *type* and $m \Vdash v \sim v \in R$. In order to show $\Gamma \vdash_m t_2 : T \circledR v \in_\alpha \Sigma(A_0, B_0)$ we merely need the following facts:
   - $\Gamma \vdash_m \mathsf{fst}(t_2) : T_0 \circledR \underline{\mathbf{fst}}(v) \in_\alpha A_0$
   - $\Gamma \vdash_m \mathsf{snd}(t_2) : T_1[\mathsf{id}.(\mathsf{fst}(t_2))] \circledR \underline{\mathbf{snd}}(v) \in_\alpha B_0[\underline{\mathbf{fst}}(v)]$

   We quickly note that $\Gamma \vdash \mathsf{fst}(t_1) = \mathsf{fst}(t_2) : T_0$ and $\Gamma \vdash \mathsf{snd}(t_1) = \mathsf{snd}(t_2) : T_1[\mathsf{id}.(\mathsf{fst}(t_1))]$ by congruence. We also have $\Gamma \vdash T_1[\mathsf{id}.(\mathsf{fst}(t_1))] = T_1[\mathsf{id}.(\mathsf{fst}(t_2))]$ *type*. We use the latter fact with Lemma 4.3.6 to conclude $\Gamma \vdash_m \mathsf{snd}(t_1) : T_1[\mathsf{id}.(\mathsf{fst}(t_2))] \circledR \underline{\mathbf{snd}}(v) \in_\alpha B_0[\underline{\mathbf{fst}}(v)]$. We already have by assumption that $\Gamma \vdash_n \mathsf{fst}(t) : T_0 \circledR \underline{\mathbf{fst}}(v) \in_\alpha A_0$.
   The conclusion then follows from $\sigma \models_m A_0 \sim A_1$ and $\sigma \models_m B_0[\underline{\mathbf{fst}}(v)] \sim B_1[\underline{\mathbf{fst}}(v)]$.

Case.

$$\frac{\forall m.\ \sigma \models_m A_0 \sim A_1 \downarrow S(m) \qquad R \Vdash v_0 \sim v_1 \in n \iff \forall m.\ S(m) \Vdash \underline{\mathbf{open}}(v_0) \sim \underline{\mathbf{open}}(v_1) \in m}{\mathbf{Box}[\sigma] \models_n \Box A_0 \sim \Box A_1 \downarrow R}$$

We wish to show $\sigma \models_n \Box A_0 \sim \Box A_1 \downarrow R$.

First we observe that from $\sigma \models_n A_0 \sim A_1 \downarrow S(m)$ we may conclude $\tau_\alpha \models_n A_0 \sim A_1 \downarrow S(m)$ and so $\tau_\alpha \models_n \Box A_0 \sim \Box A_1 \downarrow R$ holds.

Second, we wish to show that if $m \leq n$ and $\Gamma \vdash_m t_1 : T \circledR v \in_\alpha \Box A_0$ such that $\Gamma \vdash t_1 = t_2$ *typeT* that $\Gamma \vdash_m t_2 : T \circledR v \in_\alpha \Box A_1$ holds. We unfold $\Gamma \vdash_m t_1 : T \circledR v \in_\alpha \Box A_0$:

- $n \Vdash v \sim v \in R$ and $\Gamma \vdash t : T$;
- $\Gamma \vdash T = \Box T'$ *type* for some T'
- for all $m, \Gamma.\text{🔒} \vdash_m [t]_{\text{🔓}} : T' \circledR \underline{\mathbf{open}}(v) \in_\alpha \Box A_0$

We wish to show $\Gamma.\text{🔒} \vdash_m t_2 : T \circledR v \in_\alpha \Box A_0$. First, from our assumption we have some T' such that $\Gamma \vdash T = \Box T'$ *type* and $\Gamma \vdash t_1 : T$ as well as $m \Vdash v \sim v \in R$. We therefore just need to show that for all m' that $\Gamma.\text{🔒} \vdash_{m'} [t_2]_{\text{🔓}} : T' \circledR \underline{\mathbf{open}}(v) \in_\alpha A_0$ holds. First, we observe that we have $\sigma \models_{m'} A_0 \sim A_1 \downarrow S(m)$ by assumption. Furthermore, by congruence we have $\Gamma.\text{🔒} \vdash [t_1]_{\text{🔓}} = [t_2]_{\text{🔓}} : T'$. Therefore, since $\Gamma.\text{🔒} \vdash_{n'} [t_1]_{\text{🔓}} : T' \circledR \underline{\mathbf{open}}(v) \in_\alpha A_0$ we're done.

Case.

$$\frac{e_0 \sim e_1 \in \mathcal{N}e \qquad R = \{(m, \uparrow^{B_0} e_0, \uparrow^{B_1} e_1) \mid e_0 \sim e_1 \in \mathcal{N}e\}}{\mathbf{Ne} \models_n \uparrow^{A_0} e_0 \sim \uparrow^{A_1} e_1 \downarrow R}$$

Immediate by transitivity of = on terms.

Case.

$$\frac{j < \alpha}{\mathbf{Univ}_\alpha \models_n \mathbf{U}_j \sim \mathbf{U}_j \downarrow \{(m, A_0, A_1) \mid \tau_j \models_m A_0 \sim A_1\}}$$

Immediate by Lemma 4.3.6.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad n \Vdash v_0 \sim u_0 \in R \qquad n \Vdash v_1 \sim u_1 \in R}{\mathbf{Id}[\sigma] \models_n \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)}$$

Immediate by transitivity of = on terms.

Case.

$$\frac{}{\mathbf{Nat} \models_n \mathbf{nat} \sim \mathbf{nat} \downarrow [\![\mathbb{N}]\!]}$$

Immediate by transitivity of = on terms. □

Lemma 4.3.8. *If $\beta \leq \alpha$ and $\tau_\beta \models_n A \sim A$ then the following holds:*

1. *If $\Gamma \vdash_n T ® A\ \mathsf{type}_\beta$ if and only if $\Gamma \vdash_n T ® A\ \mathsf{type}_\alpha$.*
2. *If $\Gamma \vdash_n t : T ® v \in_\beta A$ if and only if $\Gamma \vdash_n t : T ® v \in_\alpha A$.*

Proof. In order to do this we show that the following is a pre-fixed point of $\mathbf{Types}_\beta$:

$$\frac{\begin{array}{c}\tau_\beta \models_{A_0} A_1 \sim R \qquad (\forall m \leq n, \Gamma, T.\ \Gamma \vdash_m T ® A\ \mathsf{type}_\beta \iff \Gamma \vdash_m T ® A\ \mathsf{type}_\alpha) \\ (\forall m \leq n, \Gamma, v, t, T.\ \Gamma \vdash_m t : T ® v \in_\beta A \iff \Gamma \vdash_n t : T ® v \in_\alpha A)\end{array}}{\sigma \models_{A_0} A_1 \sim R}$$

All cases are straightforward except the case for $\mathbf{Univ}_\beta$. Therefore we only show this case.

Case.

$$\frac{j < \beta}{\mathbf{Univ}_\beta \models_n \mathbf{U}_j \sim \mathbf{U}_j \downarrow \{(m, A_0, A_1) \mid \tau_j \models_m A_0 \sim A_1\}}$$

In this case we have some $j < \beta$ and so $j < \alpha$. We set $R = \{(m, A_0, A_1) \mid \tau_j \models_m A_0 \sim A_1\}$.

We observe that $\tau_\beta \models_n \mathbf{U}_j \sim \mathbf{U}_j \downarrow R$ as τ_β is closed under $\mathbf{Univ}_\beta$.

Next, observe that $\Gamma \vdash_m T ® \mathbf{U}_j\ \mathsf{type}_\alpha$ if and only if $\Gamma \vdash T = \mathsf{U}_j$ *type*. However, we also have that $\Gamma \vdash_m T ® \mathbf{U}_j\ \mathsf{type}_\beta$ holds if and only if $\Gamma \vdash T = \mathsf{U}_j$ *type* holds.

Moreover, if we have some $m \leq n, \Gamma, t, T$, and v such that $\Gamma \vdash_m t : T ® v \in_\beta \mathbf{U}_j$ then that the following conditions hold:

- $n \Vdash v \sim v \in R$;
- $\Gamma \vdash t : T$ and $\Gamma \vdash T = \mathsf{U}_i$ *type*;
- $\Gamma \vdash_n t ® v\ \mathsf{type}_i$.

These, however, is precisely equivalent the definition of $\Gamma \vdash_m t : T ® v \in_\alpha \mathbf{U}_j$ as $\alpha \geq \beta$. □

Lemma 4.3.9. *If $\Gamma \vdash_n t : T ® v \in_\alpha A$ then $\Gamma \vdash_n T ® A\ \mathsf{type}_\alpha$.*

Proof. In order to show this we proceed by induction on (α, A, n). We proceed by case on $\Gamma \vdash_n t : T ® v \in_\alpha A$. All cases are trivial, however, as we have added all appropriate extra premises to $\Gamma \vdash_n t : T ® v \in_\alpha A$ to ensure that this fact holds. □

We now prove the "compatibility" lemma telling us how what it means for a term and value to be connected by this logical relation. This is the equivalent of Lemma 3.2.7.

Lemma 4.3.10 (Compatibility with quotation). *If $\Gamma \vdash T$ type and for all $r : \Gamma' \leq \Gamma$ if we have some T' such that $\lceil e \rceil_{\|\Gamma'\|} = T'$ and $\Gamma' \vdash T[r] = T'$ type then $\Gamma \vdash_n T \circledR \uparrow^{-} e$ type_α.*

Proof. Suppose that we have $\Gamma \vdash T$ *type* such that for all $r : \Gamma' \leq \Gamma$ and $\lceil e \rceil_{\|\Gamma'\|} = T'$ and $\Gamma' \vdash T[r] = T'$ *type*. We wish to show $\Gamma \vdash_n T \circledR \uparrow^{-} e$ type_α but this is immediate by definition. ◻

Lemma 4.3.11 (Compatibility with quotation). *The following three facts hold for any n, α, and A such that $\tau_\alpha \models_n A \sim A$.*

1. *If $\Gamma \vdash_n T \circledR A$ type_α then for all $r : \Gamma' \leq \Gamma$, there is some T' such that $\lceil A \rceil^{\mathsf{ty}}_{\|\Gamma'\|} = T'$ and $\Gamma' \vdash T[r] = T'$ type.*
2. *If $\Gamma \vdash_n t : T \circledR v \in_\alpha A$ then for all $r : \Gamma' \leq \Gamma$ we have $\lceil \downarrow^{A} v \rceil_{\|\Gamma'\|} = t'$ and $\Gamma' \vdash t[r] = t' : T[r]$.*
3. *If $\Gamma \vdash_n T \circledR A$ type_α and $\Gamma \vdash t : T$ and if for some e we have for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$ then $\Gamma \vdash_n t : T \circledR \uparrow^{A} e \in_\alpha A$.*

Proof. We start by induction on α. We then prove these facts by together by showing $\sigma \models_n A_0 \sim A_1 \downarrow R$ is a pre-fixed point. Let $\sigma \models_n A_0 \sim A_1 \downarrow R$ hold if and only if the following conditions hold:

- $\tau_\alpha \models_n A_0 \sim A_1 \downarrow R$;
- For all $m \leq n$ and $\Gamma \vdash_m T \circledR A$ type_α there exists T' such that $\lceil A \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T = T'$ *type*;
- For all $m \leq n$ and $\Gamma \vdash_m t : T \circledR v \in_\alpha A$ there exists t' such that $\lceil \downarrow^{A} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma' \vdash t = t' : T$;
- For all $m \leq n$, $\Gamma \vdash_m T \circledR A$ type_α, $\Gamma \vdash t : T$, and if for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ and $\Gamma' \vdash t[r] = t' : T[r]$ then $\Gamma \vdash_m t : T \circledR \uparrow^{A} e \in_\alpha A$.

Suppose that $\mathbf{Types}_\alpha[\sigma] \models_n A_0 \sim A_1 \downarrow R$. We wish to show $\sigma \models_n A_0 \sim A_1 \downarrow R$.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R_0 \qquad \sigma \models_n R_0 \gg B_0 \sim B_1 \downarrow R_1 \qquad R = [\![\Pi]\!](R_0, R_1)}{\mathbf{Pi}[\sigma] \models_n \Pi(A_0, B_0) \sim \Pi(A_1, B_1) \downarrow R}$$

We wish to show $\sigma \models_n \Pi(A_0, B_0) \sim \Pi(A_1, B_1) \downarrow R$. We observe that $\sigma \leq \tau_\alpha$ and so we have $\mathbf{Pi}[\tau_\alpha] \models_n \Pi(A_0, B_0) \sim \Pi(A_1, B_1) \downarrow R$. From the definition of τ_α we then have $\tau_\alpha \models_n \Pi(A_0, B_0) \sim \Pi(A_1, B_1) \downarrow R$. Therefore, we must show three more facts:

Subgoal.

For any $m \leq n$, Γ, T, if $\Gamma \vdash_m T \circledR A$ type_α then there is some T' such that $\lceil \Pi(A_0, B_0) \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T = T'$ type.

Suppose we have $m \leq n$, Γ, T, $\Gamma \vdash_m T \circledR \Pi(A_0, B_0)$ type_α. We wish to show that there is some T' such that $\lceil \Pi(A_0, B_0) \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T = T'$ *type*.
First, we observe by inversion that there is some T_0 and T_1 such that $\Gamma \vdash T = \Pi(T_0, T_1)$ *type*. Furthermore, we must have $\Gamma \vdash_m T_0 \circledR A_0$ type_α. Finally, for any $m' \leq m$ and $r : \Gamma' \leq \Gamma$ we have that if $\Gamma' \vdash_{m'} t : T_0[r] \circledR v \in_\alpha A_0$ then $\Gamma' \vdash_{m'} T_1[r.t] \circledR B_0[v]$ type_α.
First, $\sigma \models_m A_0 \sim A_0$ tells us that there exists some T_0' such that $\lceil A_0 \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T_0'$ and $\Gamma \vdash T_0 = T_0'$ *type*.
Next, again by $\sigma \models_m A_0 \sim A_0$ we deduce that in the context $\Gamma.T_1$ that the following holds:

$$\Gamma.T_0 \vdash_m \mathsf{var}_0 : T_0[\mathsf{p}^1] \circledR \uparrow^{A_0} \mathsf{var}_{\|\Gamma\|} \in_\alpha A_0$$

We also observe that there is an r, p^1, such that $r : \Gamma.T_0 \le \Gamma$. Therefore, we may use our induction hypothesis to conclude the following:

$$\Gamma.T_0 \vdash_m T_1[r.\mathsf{var}_0] \circledR B_0[\uparrow^{A_0} \mathbf{var}_{\|\Gamma\|}] \ \mathsf{type}_\alpha$$

Moreover, since we have $m \Vdash \uparrow^{A_0} \mathbf{var}_{\|\Gamma\|} \sim \uparrow^{A_0} \mathbf{var}_{\|\Gamma\|} \in R_0$ we therefore we have a relation:

$$\sigma \models_m B_0[\uparrow^{A_0} \mathbf{var}_{\|\Gamma\|}] \sim B_1[\uparrow^{A_0} \mathbf{var}_{\|\Gamma\|}] \downarrow R_2(\uparrow^{A_0} \mathbf{var}_{\|\Gamma\|}, \uparrow^{A_0} \mathbf{var}_{\|\Gamma\|})$$

Then, by definition of σ we have that there is some T_1' such that $\lceil B_0[\uparrow^{A_0} \mathbf{var}_{\|\Gamma\|}]\rceil^{\mathsf{ty}}_{\|\Gamma.T_0\|} = T_1'$ and $\Gamma.T_0 \vdash T_1[r.\mathsf{var}_0] = T_1'$ *type*. We know that $\Gamma.T_0 \vdash r.\mathsf{var}_0 = \mathsf{id} : \Gamma.T_0$ as $r = \mathsf{p}^1$ and so $\Gamma.T_0 \vdash T_1 = T_1'$ *type* by transitivity.

However, by inspection on the definition of quotation this tells us that $\lceil \mathbf{\Pi}(A_0, B_0)\rceil^{\mathsf{ty}}_{\|\Gamma\|} = \Pi(T_0', T_1')$ and $\Gamma \vdash \Pi(T_0, T_1) = \Pi(T_0', T_1')$ *type* by congruence.

Subgoal.

> *For any $m \le n$, Γ, t, T, v, if $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{\Pi}(A_0, B_0)$ then we have $\lceil \downarrow^{\mathbf{\Pi}(A_0,B_0)} v\rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.*

Suppose we have $m \le n$, Γ, t, T, and v such that $\Gamma \vdash_m t : T \circledR v \in_\alpha \mathbf{\Pi}(A_0, B_0)$.

We wish to show $\lceil \downarrow^{\mathbf{\Pi}(A_0,B_0)} v\rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.

First, we invert upon $\Gamma \vdash_m t : T \circledR v \in_\alpha A$ to determine that there must be some T_0 and T_1 such that $\Gamma \vdash T = \Pi(T_0, T_1)$ *type*, $\Gamma \vdash t : T$, and $m \Vdash v \sim v \in R$. We have $\Gamma \vdash_m T_0 \circledR A_0 \ \mathsf{type}_\alpha$. We also have that for any $m' \le m$ and $r : \Gamma' \le \Gamma$ that if $\Gamma' \vdash_{m'} t' : T_0[r] \circledR a \in_\alpha A_0$ then $\Gamma' \vdash_{m'} t[r](t') : T_1[r.t'] \circledR \underline{\mathbf{app}}(v, a) \in_\alpha B_0[a]$.

Now, from our assumption that $\Gamma \vdash_m T_0 \circledR A_0 \ \mathsf{type}_\alpha$ and monotonicity, we have that $\Gamma.T_0 \vdash_m T_0[\mathsf{p}^1] \circledR A_0 \ \mathsf{type}_\alpha$. We may then use $\sigma \models_m A_0 \sim A_0$ to conclude that $\Gamma.T_0 \vdash_m \mathsf{var}_0 : T_0[\mathsf{p}^1] \circledR \uparrow^{A_0} \mathbf{var}_{\|\Gamma\|} \in_\alpha A_0$.

We may use this fact to conclude the following:

$$\Gamma.T_0 \vdash_m t[\mathsf{p}^1](\mathsf{var}_0) : T_1[\mathsf{p}^1.\mathsf{var}_0] \circledR \underline{\mathbf{app}}(v, \uparrow^{A_0} \mathbf{var}_{\|\Gamma\|}) \in_\alpha B_0[\uparrow^{A_0} \mathbf{var}_{\|\Gamma\|}]$$

By closure under $=$ we may simplify this:

$$\Gamma.T_0 \vdash_m t[\mathsf{p}^1](\mathsf{var}_0) : T_2 \circledR \underline{\mathbf{app}}(v, \uparrow^{A_0} \mathbf{var}_{\|\Gamma\|}) \in_\alpha B_0[\uparrow^{A_0} \mathbf{var}_{\|\Gamma\|}]$$

Now, we know that $m \Vdash \uparrow^{A_0} \mathbf{var}_{\|\Gamma\|} \sim \uparrow^{A_0} \mathbf{var}_{\|\Gamma\|} \in R_0$ either from Lemma 3.2.7 or from Lemma 4.2.1. We then have $\sigma \models_m B_0[\uparrow^{A_0} \mathbf{var}_{\|\Gamma\|}] \sim B_0[\uparrow^{A_0} \mathbf{var}_{\|\Gamma\|}]$ and so we may conclude that there is some t' such that the following two conditions hold:

$$\lceil \downarrow^{B_0[\uparrow^{A_0}\mathbf{var}_{\|\Gamma\|}]} \underline{\mathbf{app}}(v, \uparrow^{A_0} \mathbf{var}_{\|\Gamma\|})\rceil_{\|\Gamma\|+1} = t'$$
$$\Gamma.T_0 \vdash t[\mathsf{p}^1](\mathsf{var}_0) = t' : T_1$$

But, we then have that $\lceil \downarrow^{\mathbf{\Pi}(A_0,B_0)} v\rceil_{\|\Gamma\|} = \lambda t'$ and $\Gamma \vdash t = \lambda t' : \Pi(T_0, T_1)$ by eta and congruence.

Subgoal.

> *For any $m \le n$, Γ, t, T, if $\Gamma \vdash_m T \circledR \mathbf{\Pi}(A_0, B_0)$ type_α, $\Gamma \vdash t : T$, and if for some e we have for all $r : \Gamma' \le \Gamma$ we have $\lceil e\rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$ then $\Gamma \vdash_m t : T \circledR \uparrow^A e \in_\alpha \mathbf{\Pi}(A_0, B_0)$.*

Suppose we have $m \leq n, \Gamma, t, T$, and e such $\Gamma \vdash_m T ® \Pi(A_0, B_0)$ type_α and if for some e we have for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$.

We wish to show that $\Gamma \vdash_m t : T ® \uparrow^{\Pi(A_0,B_0)} e \in_\alpha \Pi(A_0, B_0)$.

First, we invert on $\Gamma \vdash_m T ® A$ type_α to conclude that there is some $\Gamma \vdash \Pi(T_0, T_1) = T$ *type* such that $\Gamma \vdash_m T_0 ® A_0$ type_α. We must have that if $m' \leq m$ and $r : \Gamma' \leq \Gamma$ such that $\Gamma' \vdash_{m'} t' : T_0[r] ® v \in_\alpha A_0$ then we have $\Gamma' \vdash_{m'} T_1[r.t'] ® B_0[v]$ type_α.

We wish to show $\Gamma \vdash_m t : T ® \uparrow^{\Pi(A_0,B_0)} e \in_\alpha A$.

We merely need to show that if we have some $m' \leq m$ and $r : \Gamma' \leq \Gamma$ such that $\Gamma' \vdash_{m'} t' : T_0[r] ® a \in_\alpha A_0$ then the following holds:

$$\Gamma' \vdash_{m'} t[r](t') : T_1[r.t'] ® \underline{\mathbf{app}}(\uparrow^{A_0} e, a) \in_\alpha B_0[a]$$

Observe that $\underline{\mathbf{app}}(\uparrow^{\Pi(A_0,B_0)} e, a) = \uparrow^{B_0[a]} e.\mathbf{app}(\downarrow^{A_0} a)$ and $B_0[a]$ is defined from our assumption of $\sigma \models_{m'} R_0 \gg B_0 \sim B_0$ holds and since $m \Vdash a \sim a \in R_0$ by Lemma 4.2.1. Since $e.\mathbf{app}(\downarrow^{A_0} a)$ is a neutral so we will apply our induction hypothesis.

First, we have that for all $r' : \Gamma'' \leq \Gamma'$ that $\lceil \downarrow^{A_1} a \rceil_{\|\Gamma'\|} = t_a$ for some t_a such that $\Gamma'' \vdash t'[r'] = t_a : T_0[r \circ r']$ from our induction hypothesis.

Now, we had by assumption that $r' : \Gamma'' \leq \Gamma'$ $\lceil e \rceil_{\|\Gamma'\|} = t_f$ for some t_f such that $\Gamma'' \vdash t[r \circ r'] = t_a : T_0[r \circ r']$. We have made use the functoriality of explicit substitutions here along with the transitivity of definitional equality.

Now finally, this tells us that for any $r' : \Gamma'' \leq \Gamma'$ that $\lceil e.\mathbf{app}(\downarrow^{A_0} a) \rceil_{\|\Gamma'\|} = t_t$ such that $\Gamma'' \vdash t[r](t')[r'] = t_t : T_1[(r.t') \circ r']$. We may then use the fact that $\sigma \models_{m'} B_0[a] \sim B_1[a]$ to conclude that $\Gamma' \vdash_{m'} t : T ® \uparrow^{\Pi(A_0,B_0)} e \in_\alpha \Pi(A_0, B_0)$ as required.

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R_0 \qquad \sigma \models_n R_0 \gg B_0 \sim B_1 \downarrow R_1 \qquad R = [\![\Sigma]\!](R_0, R_1)}{\mathrm{Sg}[\sigma] \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow R}$$

We wish to show $\sigma \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow R$. We observe that $\sigma \leq \tau_\alpha$ and so we have $\mathrm{Sg}[\sigma] \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow R$. By definition of τ_α we have $\tau_\alpha \models_n \Sigma(A_0, B_0) \sim \Sigma(A_1, B_1) \downarrow R$.

Therefore, we must show three more facts:

Subgoal.

For any $m \leq n, \Gamma, T$, if $\Gamma \vdash_m T ® \Sigma(A_0, B_0)$ type_α then there is some T' such that $\lceil A \rceil^{\mathrm{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T = T'$ type.

Identical to case for $\Pi(-, -)$.

Subgoal.

For any $m \leq n, \Gamma, t, T, v$, if $\Gamma \vdash_m T ® \Sigma(A_0, B_0)$ type_α and $\Gamma \vdash_m t : T ® v \in_\alpha \Sigma(A_0, B_0)$ then we have $\lceil \downarrow^{\Sigma(A_0,B_0)} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.

For this, suppose we have $m \leq n, \Gamma, t, T$, and v. If we have $\Gamma \vdash_m t : T ® v \in_\alpha \Sigma(A_0, B_0)$ then we wish to show $\lceil \downarrow^{\Sigma(A_0,B_0)} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.

First, we perform inversion on $\Gamma \vdash_m t : T ® v \in_\alpha \Sigma(A_0, B_0)$. This tells us that the following facts hold:

- $m \Vdash v \sim v \in R$ and $\Gamma \vdash t : T$;
- $\Gamma \vdash T = \Sigma(T_0, T_1)$ *type* for some T_0, T_1;
- if $m' \leq m$ and $r : \Gamma' \leq \Gamma$, then $\Gamma' \vdash_{m'} t' : T_0[r] ® a \in_\alpha A_0$ implies $\Gamma' \vdash_{m'} T_1[r.t'] ® B_0[a]$ type_α;

- $\Gamma \vdash_m \mathsf{fst}(t) : T_0 \circledR \underline{\mathbf{fst}}(v) \in_\alpha A_0$;
- $\Gamma \vdash_m \mathsf{snd}(t) : T_1[\mathsf{id}.(\mathsf{fst}(t))] \circledR \underline{\mathbf{snd}}(v) \in_\alpha B_0[\underline{\mathbf{fst}}(v)]$.

Now, we have $\sigma \models_n A_0 \sim A_0$ and so $\lceil \downarrow^{A_0} \underline{\mathbf{fst}}(v) \rceil_{\|\Gamma\|} = t_f$ such that $\Gamma \vdash \mathsf{fst}(t) = t_f : T_0$. Furthermore, since $m \Vdash \underline{\mathbf{fst}}(v) \sim \underline{\mathbf{fst}}(v) \in R_0$ we must have $\sigma \models_m B_0[\underline{\mathbf{fst}}(v)] \sim B_0[\underline{\mathbf{fst}}(v)]$. Therefore, we may conclude that $\lceil \downarrow^{B_0[\underline{\mathbf{fst}}(v)]} \underline{\mathbf{snd}}(v) \rceil_{\|\Gamma\|} = t_s$ such that $\Gamma \vdash \mathsf{snd}(t) = t_s : T_1[\mathsf{id}.(\mathsf{fst}(t))]$.

Now from these two facts, we have $\lceil \downarrow^{\Sigma(A_0,B_0)} v \rceil_{\|\Gamma\|} = (t_f, t_s)$ and so $\Gamma \vdash t = (t_f, t_s) : T$ by congruence and eta.

Subgoal.

> *For any $m \leq n, \Gamma, t, T$, if $\Gamma \vdash t : T$ and $\Gamma \vdash_m T \circledR \Sigma(A_0, B_0)\ \mathsf{type}_\alpha$ and if for some e we have for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$ then $\Gamma \vdash_m t : T \circledR \uparrow^{\Sigma(A_0,B_0)} e \in_\alpha \Sigma(A_0, B_0)$.*

Suppose we have some $m \leq n, \Gamma, \Gamma \vdash t : T$ and that $\Gamma \vdash_m T \circledR \Sigma(A_0, B_0)\ \mathsf{type}_\alpha$. Suppose further that there is some e such that for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ and $\Gamma' \vdash t[r] = t' : T[r]$. We wish to show that $\Gamma \vdash_m t : T \circledR \uparrow^{\Sigma(A_0,B_0)} e \in_\alpha \Sigma(A_0, B_0)$.

First, we observe by inversion that that we must have $\Gamma \vdash T = \Sigma(T_0, T_1)$ *type* such that $\Gamma \vdash_m T_0 \circledR A_0\ \mathsf{type}_\alpha$ and for all $\Gamma \vdash_m t_1 : T_0 \circledR v_f \in_\alpha A_0$ we also have $\Gamma \vdash_m T_1[\mathsf{id}.t_1] \circledR B_0[v_f]\ \mathsf{type}_\alpha$.

Now, in order to show $\Gamma \vdash_m t : T \circledR \uparrow^{\Sigma(A_0,B_0)} e \in_\alpha \Sigma(A_0, B_0)$ it suffices to show the following two facts:

$$\Gamma \vdash_m \mathsf{fst}(t) : T_0 \circledR \underline{\mathbf{fst}}(\uparrow^{\Sigma(A_0,B_0)} e) \in_\alpha A_0$$
$$\Gamma \vdash_m \mathsf{snd}(t) : T_1[\mathsf{id}.(\mathsf{fst}(t))] \circledR \underline{\mathbf{snd}}(\uparrow^{\Sigma(A_0,B_0)} e) \in_\alpha B_0[\underline{\mathbf{fst}}(\uparrow^{\Sigma(A_0,B_0)} e)]$$

We show the first by observing that $\underline{\mathbf{fst}}(\uparrow^{\Sigma(A_0,B_0)} e) = \uparrow^{A_0} e.\mathbf{fst}$ so it suffices to show that for any $r : \Gamma' \leq \Gamma$ we have $\lceil e.\mathbf{fst} \rceil_{\|\Gamma'\|} = t'$ and $\Gamma' \vdash (\mathsf{fst}(t))[r] = t' : T_0[r]$. This conclusion is immediate by the definition of quotation and our assumption that this holds for t and e.

We then have that $\Gamma \vdash_m T_1[\mathsf{id}.(\mathsf{fst}(t))] \circledR B_0[\uparrow^{A_0} e.\mathbf{fst}]\ \mathsf{type}_\alpha$. Therefore, $B_0[\uparrow^{A_0} e.\mathbf{fst}]$ terminates and so $\underline{\mathbf{snd}}(\uparrow^{\Sigma(A_0,B_0)} e) = \uparrow^{B_0[\uparrow^{A_0} e.\mathbf{fst}]} e$.

In order to show the second part, then, it suffices to show $r : \Gamma' \leq \Gamma$ we have $\lceil e.\mathbf{snd} \rceil_{\|\Gamma'\|} = t'$ and $\Gamma' \vdash \mathsf{snd}(t)[r] = t' : T_2[(\mathsf{id}.(\mathsf{fst}(t))) \circ r]$.

This is similar to the case for the first projection: it follows from the definition of quotation and our assumption that this holds for t and e.

Case.

$$\frac{\forall m.\ \sigma \models_m A_0 \sim A_1 \downarrow S(m) \qquad R = [\![\Box]\!](S)}{\mathbf{Box}[\sigma] \models_n \Box A_0 \sim \Box A_1 \downarrow R}$$

We wish to show $\sigma \models_n \Box A_0 \sim \Box A_1 \downarrow R$. We observe that $\sigma \leq \tau_\alpha$ and so we have $\mathbf{Box}[\sigma] \models_n \Box A_0 \sim \Box A_1 \downarrow R$. Therefore, we may conclude $\tau_\alpha \models_n \Box A_0 \sim \Box A_1 \downarrow R$. Therefore, we must show three more facts:

Subgoal.

> *For any $m \leq n, \Gamma, T$, if $\Gamma \vdash_m T \circledR \Box A_0\ \mathsf{type}_\alpha$ then there is some T' such that $\lceil \Box A_0 \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T = T'$ type.*

Suppose we have $m \leq n, \Gamma, T$ and $\Gamma \vdash_m T \circledR \Box A_0\ \mathsf{type}_\alpha$. We wish to show that we have some T' such that $\lceil \Box A_0 \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T = T'$ *type*.

First, we invert upon on $\Gamma \vdash_m T Ⓡ \Box A_0\ \mathsf{type}_\alpha$, we then must have that $\Gamma \vdash T = \Box T'$ *type* and for all m, we have $\Gamma.🔒 \vdash_m T' Ⓡ A_0\ \mathsf{type}_\alpha$. Since $\sigma \models_m A_0 \sim A_1$ we may then use the latter fact to conclude that $\lceil A_0 \rceil^{\mathsf{ty}}_{\|\Gamma\|} = S$ such that $\Gamma \vdash T' = S$ *type*. By definition, we must have $\lceil \Box A_0 \rceil^{\mathsf{ty}}_{\|\Gamma\|} = \Box S$. Finally, $\Gamma \vdash T = \Box S$ *type* by transitivity and congruence.

Subgoal.

> *For any $m \leq n, \Gamma, t, T, v$, if $\Gamma \vdash_m t : T Ⓡ v \in_\alpha \Box A_0$ then we have $\lceil \downarrow^{A} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.*

For this, suppose we have $m \leq n, \Gamma, t, T, v$ such that $\Gamma \vdash_m t : T Ⓡ v \in_\alpha \Box A_0$. We wish to show that the following holds: $\lceil \downarrow^{\Box A_0} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.

We first perform inversion on $\Gamma \vdash_m t : T Ⓡ v \in_\alpha \Box A_0$. We then have the following facts:

- $m \Vdash v \sim v \in R$ and $\Gamma \vdash t : T$;
- $\Gamma \vdash T = \Box T'$ *type* for some T'
- for all m, $\Gamma.🔒 \vdash_m [t]_{🔓} : T' Ⓡ \underline{\mathbf{open}}(v) \in_\alpha A_0$

We have $\sigma \models_m A_0 \sim A_0$ by assumption, so from $\Gamma.🔒 \vdash_m [t]_{🔓} : T' Ⓡ \underline{\mathbf{open}}(v) \in_\alpha A_0$ we may conclude that there is some t' such that $\lceil \downarrow^{A_0} \underline{\mathbf{open}}(v) \rceil_{\|\Gamma.🔒\|} = t'$ such that $\Gamma.🔒 \vdash [t]_{🔓} = t' : T'$. By definition of quotation then, we have that $\lceil \downarrow^{\Box A} v \rceil_{\|\Gamma\|} = [t']_{🔒}$ and by congruence we have $\Gamma \vdash [[t]_{🔓}]_{🔒} = [t']_{🔒} : \Box T'$.

Subgoal.

> *For any $m \leq n, \Gamma, t, T$, if $\Gamma \vdash t : T$ and $\Gamma \vdash_m T Ⓡ \Box A_0\ \mathsf{type}_\alpha$ and if for some e we have for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$ then $\Gamma \vdash_m t : T Ⓡ \uparrow^{\Box A_0} e \in_\alpha \Box A_0$.*

Suppose we have $m \leq n, \Gamma, t, T$ such that $\Gamma \vdash t : T$ and $\Gamma \vdash_m T Ⓡ \Box A_0\ \mathsf{type}_\alpha$. Furthermore, suppose we have e we have for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$. We wish to show $\Gamma \vdash_m t : T Ⓡ \uparrow^{\Box A_0} e \in_\alpha A$.

We start by performing inversion on $\Gamma \vdash_m T Ⓡ A\ \mathsf{type}_\alpha$. This tells us that there is some T' such that $\Gamma \vdash T = \Box T'$ *type* and for all m' we have $\Gamma.🔒 \vdash_{m'} T' Ⓡ A'\ \mathsf{type}_\alpha$.

We also observe that for any $r : \Gamma' \leq \Gamma.🔒$ we have $r : \Gamma'^{🔓} \leq \Gamma$ by Lemma 1.2.11. Therefore, we have $\lceil e.\mathbf{open} \rceil_{\|\Gamma'\|} = [t']_{🔓}$ where $\lceil e \rceil_{\|\Gamma'^{🔓}\|} = t'$ such that $\Gamma' \vdash ([t]_{🔓})[r] = t' : T'[r]$ from our assumption about quotation.

Next, observe that $e.\mathbf{open}$ is a neutral. From our prior assumptions then we have that $\Gamma.🔒 \vdash_{m'} [t]_{🔓} : T' Ⓡ \uparrow^{A_0} e.\mathbf{open} \in_\alpha A_0$. This is sufficient to give us the goal.

Case.

$$\frac{R = \{(m, \uparrow^{B_0} e_0, \uparrow^{B_1} e_1) \mid e_0 \sim e_1 \in \mathcal{N}e\}}{\mathbf{Ne} \models_n \uparrow^{-} e_0 \sim \uparrow^{-} e_1 \downarrow R}$$

We must show $\sigma \models_n \uparrow^{-} e_0 \sim \uparrow^{-} e_1 \downarrow R$. We therefore immediately have $\tau_\alpha \models_n \uparrow^{-} e_0 \sim \uparrow^{-} e_1 \downarrow R$. We just need to show three facts then.

Subgoal.

> *For any $m \leq n, \Gamma, T$, if $\Gamma \vdash_m T Ⓡ \uparrow^{-} e_0\ \mathsf{type}_\alpha$ then there is some T' such that $\lceil \uparrow^{-} e_0 \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T = T'$ type.*

Suppose we have $m \leq n, \Gamma, T$ and $\Gamma \vdash_m T Ⓡ \uparrow^{-} e_0\ \mathsf{type}_\alpha$. We wish to show that we have some T' such that $\lceil \uparrow^{-} e_0 \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T = T'$ *type*. By inversion on $\Gamma \vdash_m T Ⓡ \uparrow^{-} e_0\ \mathsf{type}_\alpha$ we have that $\lceil e_0 \rceil_{\|\Gamma\|} = T'$ and $\Gamma \vdash T[\mathsf{id}] = T'$ *type* completing the proof.

Subgoal.

For any $m \leq n, \Gamma, t, T, v$, if $\Gamma \vdash_m t : T$ ® $v \in_\alpha \uparrow^- e_0$ then we have $\lceil \downarrow^{\uparrow^- e_0} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.

For this, suppose we have $m \leq n, \Gamma, t, T, v$ such that $\Gamma \vdash_m t : T$ ® $v \in_\alpha \uparrow^- e_0$. We wish to show that the following holds: $\lceil \downarrow^{\uparrow^- e_0} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.

In this case, we have by inversion that $v = \uparrow^- e$ such that $\lceil e \rceil_{\|\Gamma\|} = t'$ such that $\Gamma \vdash t[\mathrm{id}] = t' : T[\mathrm{id}]$. Our goal follows from transitivity and conversion.

Subgoal.

For any $m \leq n, \Gamma, t, T$, if $\Gamma \vdash t : T$ and $\Gamma \vdash_m T$ ® $\uparrow^- e_0$ type_α and if for some e we have for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$ then $\Gamma \vdash_m t : T$ ® $\uparrow^{\uparrow^- e_0} e \in_\alpha A$.

Suppose we have $m \leq n, \Gamma, t, T$ such that $\Gamma \vdash t : T$ and $\Gamma \vdash_m T$ ® $\uparrow^- e_0$ type_α. Furthermore, suppose we have e we have for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$. We wish to show $\Gamma \vdash_m t : T$ ® $\uparrow^{\uparrow^- e_0} e \in_\alpha A$. This is immediate by definition.

Case.

$$\frac{}{\mathbf{Nat} \models_n \mathbf{nat} \sim \mathbf{nat} \downarrow [\![\mathbb{N}]\!]}$$

We have immediately that $\tau_\alpha \models_n \mathbf{nat} \sim \mathbf{nat} \downarrow [\![\mathbb{N}]\!]$. We must show the next three facts.

Subgoal.

For any $m \leq n, \Gamma, T$, if $\Gamma \vdash_m T$ ® $\mathbf{nat}$ type_α then there is some T' such that $\lceil \mathbf{nat} \rceil^{\mathrm{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T = \mathbf{nat}$ type.

Since $\mathbf{nat}$ and the fact that we have by inversion on $\Gamma \vdash_m T$ ® $\mathbf{nat}$ type_α that $\Gamma \vdash T = \mathrm{nat}$ *type* and so the goal follows by computation.

Subgoal.

For any $m \leq n, \Gamma, t, T, v$, if $\Gamma \vdash_m t : T$ ® $v \in_\alpha \mathbf{nat}$ then we have $\lceil \downarrow^{\mathbf{nat}} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.

For this, suppose we have $m \leq n, \Gamma, t, T, v$ such that $\Gamma \vdash_m t : T$ ® $v \in_\alpha \mathbf{nat}$. We wish to show that the following holds: $\lceil \downarrow^{\mathbf{nat}} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.

We observe that $\Gamma \vdash_m t : T$ ® $v \in_\alpha \mathbf{nat}$ is inductive so we proceed by induction. We must prove three cases.

1. In the first case we have $\Gamma \vdash T = \mathrm{nat}$ *type*, $\Gamma \vdash t = \mathrm{zero} : \mathrm{nat}$, and $v = \mathbf{zero}$. Therefore, our goal is immediate by computation.
2. In the second case we have $\Gamma \vdash T = \mathrm{nat}$ *type*, $\Gamma \vdash t = \mathrm{succ}(t') : \mathrm{nat}$, and $v = \mathbf{succ}(v')$ such that $\Gamma \vdash_m t' : T$ ® $v' \in_\alpha \mathbf{nat}$. Our induction hypothesis tells us that there is some s such that $\lceil \downarrow^{\mathbf{nat}} v' \rceil_{\|\Gamma\|} = s$ such that $\Gamma \vdash t' = s : \mathrm{nat}$. Thus, by congruence and computation we're done.
3. In the final case we have $\Gamma \vdash T = \mathrm{nat}$ *type*, $v = \uparrow^{\mathbf{nat}} e$ such that $\lceil e \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : \mathrm{nat}$. This is exactly the goal however.

Subgoal.

For any $m \leq n, \Gamma, t, T$, if $\Gamma \vdash t : T$ and $\Gamma \vdash_m T$ ® $\mathbf{nat}$ type_α and if for some e we have for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$ then $\Gamma \vdash_m t : T$ ® $\uparrow^{\mathbf{nat}} e \in_\alpha A$.

Immediate by definition of $\Gamma \vdash_m t : T$ ® $\uparrow^{\mathbf{nat}} e \in_\alpha \mathbf{nat}$

Case.

$$\frac{\sigma \models_n A_0 \sim A_1 \downarrow R \qquad n \Vdash v_0 \sim u_0 \in R \qquad n \Vdash v_1 \sim u_1 \in R}{\mathbf{Id}[\sigma] \models_n \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)}$$

We immediately have $\tau_\alpha \models_n \mathbf{Id}(A_0, v_0, v_1) \sim \mathbf{Id}(A_1, u_0, u_1) \downarrow [\![\mathsf{Id}]\!](R, u_0, u_1)$. We must show the next three facts.

Subgoal.

For any $m \leq n, \Gamma, T$, if $\Gamma \vdash_m T$ ® $\mathbf{Id}(A_0, v_0, v_1)$ type_α then there is some T' such that $\lceil \mathbf{Id}(A_0, v_0, v_1) \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T = T'$ type.

We have by inversion on $\Gamma \vdash_m T$ ® $\mathbf{Id}(A_0, v_0, v_1)$ type_α that $\Gamma \vdash T = \mathsf{Id}(T', t_0, t_1)$ *type* such that $\Gamma \vdash_m T'$ ® A_0 type_α and $\Gamma \vdash_m t_i : T'$ ® $v_i \in_\alpha A_0$. We observe that from our assumption of $\sigma \models_n A_0 \sim A_1 \downarrow R$ that there must be some T_0' such that $\lceil A_0 \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T' = T_0'$ *type*. Furthermore, we must have that $\lceil \downarrow^{A_0} v_i \rceil_{\|\Gamma\|} = t_i'$ such that $\Gamma \vdash t_i = t_i' : T'$, again from $\sigma \models_n A_0 \sim A_1 \downarrow R$. Therefore, we have $\lceil \mathbf{Id}(A_0, v_0, v_1) \rceil^{\mathsf{ty}}_{\|\Gamma\|} = \mathsf{Id}(T', t_0', t_1')$. Finally, by congruence we then have $\Gamma \vdash T = \mathsf{Id}(T', t_0', t_1')$ *type*.

Subgoal.

For any $m \leq n, \Gamma, t, T, v$, if $\Gamma \vdash_m t : T$ ® $v \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$ then we have $\lceil \downarrow^{\mathbf{Id}(A_0, v_0, v_1)} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.

For this, suppose we have $m \leq n, \Gamma, t, T, v$ such that $\Gamma \vdash_m t : T$ ® $v \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$. We wish to show that the following holds: $\lceil \downarrow^{\mathbf{Id}(A_0, v_0, v_1)} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.

We proceed by inversion on $\Gamma \vdash_m t : T$ ® $v \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$. We therefore conclude that $m \Vdash v \sim v \in [\![\mathsf{Id}]\!](R, u_0, u_1)$, $\Gamma \vdash t : T$, $\Gamma \vdash T = \mathsf{Id}(T', t_0, t_1)$ *type*, $\Gamma \vdash_m T'$ ® A_0 type_α, and $\Gamma \vdash_m t_i : T'$ ® $v_i \in_\alpha A_0$. We also have that one of the following two facts is true:

- $v = \uparrow^- e$ and when $r : \Gamma' \leq \Gamma$, then $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$.
- $\Gamma \vdash t = \mathsf{refl}(t') : T$ and $v = \mathbf{refl}(v')$ for some t', v' such that $\Gamma \vdash t' = t_i : T'$.

We proceed by cases on which fact holds. If $v = \uparrow^- e$ and when $r : \Gamma' \leq \Gamma$, then $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$ then we have the desired conclusion immediately by picking $r = \mathsf{id}$.

Instead, suppose that $\Gamma \vdash t = \mathsf{refl}(t') : T$ and $v = \mathbf{refl}(v')$ for some t', v' such that $\Gamma \vdash t' = t_i : T'$. In this case we have $m \Vdash v' \sim v_0 \in R$ as $m \Vdash \mathbf{refl}(v') \sim \mathbf{refl}(v') \in [\![\mathsf{Id}]\!](R, u_0, u_1)$ and $n \Vdash u_0 \sim v_0 \in R$. We may therefore conclude that $m \vdash_\Gamma t_0 : T'$ ® $v' \in_\alpha A_0$ from Lemma 4.3.5. By induction hypothesis, then, we have that there is some t_q such that $\lceil \downarrow^{A_0} v' \rceil_{\|\Gamma\|} = t_q$ and $\Gamma \vdash t_0 = t_q : T'$. Therefore, by transitivity of equality we have $\Gamma \vdash t' = t_q : T'$. Finally, since $\lceil \downarrow^{\mathbf{Id}(A_0, v_0, v_1)} \mathbf{refl}(v') \rceil_{\|\Gamma\|} = \mathsf{refl}(t_q)$ by definition we are done by congruence.

Subgoal.

For any $m \leq n, \Gamma, t, T$, if $\Gamma \vdash t : T$ and $\Gamma \vdash_m T$ ® $\mathbf{Id}(A_0, v_0, v_1)$ type_α and if for some e we have for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$ then $\Gamma \vdash_m t : T$ ® $\uparrow^{\mathbf{Id}(A_0, v_0, v_1)} e \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$.

This is follows immediately from the definition of $\Gamma \vdash_m t : T$ ® $\uparrow^{\mathbf{Id}(A_0, v_0, v_1)} e \in_\alpha \mathbf{Id}(A_0, v_0, v_1)$.

Case.

$$\frac{j < \alpha \qquad R = \{(m, A_0, A_1) \mid \tau_j \models_m A_0 \sim A_1\}}{\mathbf{Univ}_\alpha \models_n \mathbf{U}_j \sim \mathbf{U}_j \downarrow R}$$

We have immediately that $\tau_\alpha \models_n \mathbf{U}_j \sim \mathbf{U}_j \downarrow R$. We must show the next three facts.

Subgoal.

> *For any $m \leq n, \Gamma, T$, if $\Gamma \vdash_m T$ ® $\mathbf{U}_j$ type_α then there is some T' such that $\lceil \mathbf{U}_j \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T'$ and $\Gamma \vdash T = T'$ type.*

We have by inversion on $\Gamma \vdash_m T$ ® $\mathbf{U}_j$ type_α that $\Gamma \vdash T = \mathsf{U}_j$ *type* and so the goal follows by computation.

Subgoal.

> *For any $m \leq n, \Gamma, t, T, v$, if $\Gamma \vdash_m t : T$ ® $v \in_\alpha \mathbf{U}_j$ then we have $\lceil \downarrow^{\mathbf{U}_j} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.*

For this, suppose we have $m \leq n, \Gamma, t, T, v$ such that $\Gamma \vdash_m t : T$ ® $v \in_\alpha \mathbf{U}_j$. We wish to show that the following holds: $\lceil \downarrow^{\mathbf{U}_j} v \rceil_{\|\Gamma\|} = t'$ and $\Gamma \vdash t = t' : T$.

By inversion, we have $\Gamma \vdash t : T$, $\Gamma \vdash T = \mathsf{U}_j$ *type*, $m \Vdash v \sim v \in R$, and $\Gamma \vdash_m t$ ® v type_j. However, our induction hypothesis (recall that we had proceeded by induction on α and $j < \alpha$) applied to the last fact gives us the goal immediately.

Subgoal.

> *For any $m \leq n, \Gamma, t, T$, if $\Gamma \vdash t : T$ and $\Gamma \vdash_m T$ ® $\mathbf{U}_j$ type_α and if for some e we have for all $r : \Gamma' \leq \Gamma$ we have $\lceil e \rceil_{\|\Gamma'\|} = t'$ such that $\Gamma' \vdash t[r] = t' : T[r]$ then $\Gamma \vdash_m t : T$ ® $\uparrow^{\mathbf{U}_j} e \in_\alpha \mathbf{U}_j$.*

This is Lemma 4.3.10 after unfolding $\Gamma \vdash_m t : T$ ® $\uparrow^{\mathbf{U}_j} e \in_\alpha \mathbf{U}_j$. □

Corollary 4.3.12. *If $\Gamma \vdash_n T_0$ ® A type_α and $\Gamma \vdash_n T_1$ ® A type_α then $\Gamma \vdash T_0 = T_1$ type.*

Proof. From Lemma 4.3.11 we have that $\lceil A \rceil^{\mathsf{ty}}_{\|\Gamma\|} = T'$ such that $\Gamma \vdash T_0 = T'$ *type* and $\Gamma \vdash T_1 = T'$ *type*. Therefore, the conclusion follows from transitivity. □

4.4 Soundness

Lemma 4.4.1. *Any substitution $\Gamma \vdash \delta : \Delta.A$ is definitionally equal to a substitution of the form $\delta'.t$.*

Proof. We observe that $\Gamma \vdash \mathsf{id} \circ \delta = \delta : \Delta.A$ and thus $\Gamma \vdash (\mathsf{p}^1.\mathsf{var}_0) \circ \delta = \delta : \Delta.A$. Finally, this gives us the goal:

$$\Gamma \vdash (\mathsf{p}^1 \circ \delta).\mathsf{var}_0[\delta] = \delta : \Delta.A$$

□

Lemma 4.4.2. *If Γ ctx then $\Gamma \vdash \mathsf{id} : \Gamma^{🔓}.🔒$*

Proof. Immediate by the lifting rule. □

Before stating soundness recall that by completeness (Theorem 3.3.5) if $\Gamma \vdash T$ *type* and $n \Vdash \rho_1 = \rho_2 : \Gamma$ then $\tau_\omega \models_n [\![T]\!]_{\rho_1} \sim [\![T]\!]_{\rho_2}$.

We must also extend our logical relation to substitutions now. This defines a relation $\Delta \vdash_n \delta : \Gamma$ ® ρ. This relation is defined by induction on Γ. We shall say that $\Delta \vdash_n \delta : \Gamma$ ® ρ holds when one of the following cases apply:

- $\Delta \vdash_n \delta : \cdot$ ® $\cdot$ if $\Delta \vdash \delta : \cdot$.
- $\Delta \vdash_n \delta : \Gamma.T$ ® $\rho.v$ if:
  - $\Delta \vdash \delta = \delta'.t : \Gamma.T$ for some δ', t;
  - $\tau_\omega \models_n [\![T]\!]_\rho \sim [\![T]\!]_\rho$;
  - $\Delta \vdash_n t : T[\delta']$ ® $v \in_\omega [\![T]\!]_\rho$;
  - $\Delta \vdash_n \delta' : \Gamma$ ® ρ

- $\Delta \vdash_n \delta : \Gamma.\text{🔒} \circledR \rho$ if Δ *ctx* and there exists some m such that $\Delta^{\text{🔓}} \vdash_m \delta : \Gamma \circledR \rho$.

We now prove some facts about this definition.

Lemma 4.4.3. *$\Delta \vdash_n \delta : \Gamma \circledR \rho$ is monotone in both n and Δ (the latter with respect to weakenings).*

Proof. This is a corollary of Lemma 4.3.1. □

Lemma 4.4.4. *If $\Delta \vdash_n \delta : \Gamma \circledR \rho$ then Δ ctx.*

Proof. Follows immediately by case on Γ. □

Lemma 4.4.5. *If $\Delta \vdash_n \delta_1 : \Gamma \circledR \rho$ and $\Delta \vdash \delta_1 = \delta_2 : \Gamma$ then $\Delta \vdash_n \delta_2 : \Gamma \circledR \rho$.*

Proof. Follows immediately from the transitivity of = and by induction on Γ. □

Lemma 4.4.6. *If $\Delta \vdash_n \delta : \Gamma \circledR \rho$ then there exists an $m \leq n$ such that $\Delta^{\text{🔓}} \vdash_m \delta : \Gamma^{\text{🔓}} \circledR \rho$.*

Proof. This follows by induction on Γ.

Case.

$$\Gamma = \cdot$$

In this case we must show $\Delta^{\text{🔓}} \vdash_m \delta : \cdot \circledR \rho$ and so $\Delta \vdash \delta : \cdot$ and $\rho = \cdot$. The conclusion follows by Lemma 1.2.5.

Case.

$$\Gamma = \Gamma'.T$$

In this case we must show $\Delta^{\text{🔓}} \vdash_m \delta : \Gamma'^{\text{🔓}}.T \circledR \rho$. We start by observing that $\Delta \vdash \delta = \delta'.t : \Gamma'.T$ such that $\tau_\omega \models_n [\![T]\!]_\rho \sim [\![T]\!]_\rho$, $\Delta \vdash_n t : T[\delta'] \circledR v \in_\omega [\![T]\!]_\rho$ and $\Delta \vdash_n \delta' : \Gamma' \circledR \rho$. By induction hypothesis we have that there is some $m \leq n$ such that $\Delta^{\text{🔓}} \vdash_m \delta' : \Gamma'^{\text{🔓}} \circledR \rho$. We have $\Delta^{\text{🔓}} \vdash_m t : T[\delta'] \circledR v \in_\omega [\![T]\!]_\rho$ by Lemmas 4.3.1 and 4.3.2. We have $\tau_\omega \models_m [\![T]\!]_\rho \sim [\![T]\!]_\rho$ by Lemma 3.2.5. Finally, we have $\Delta^{\text{🔓}} \vdash \delta = \delta'.t : \Gamma'^{\text{🔓}}.T$ from Lemma 1.2.10.

Case.

$$\Gamma = \Gamma'.\text{🔒}$$

In this case we must show $\Delta^{\text{🔓}} \vdash_m \delta : \Gamma'^{\text{🔓}} \circledR \rho$. We start by observing that there is some m such that $\Delta^{\text{🔓}} \vdash_m \delta : \Gamma' \circledR \rho$ and Δ *ctx*. By Lemma 4.4.3 we may assume that $m \leq n$. Next, by induction hypothesis we have $\Delta^{\text{🔓}} \vdash_m \delta : \Gamma'^{\text{🔓}} \circledR \rho$ as required. We have $\Delta^{\text{🔓}}$ *ctx* from Lemma 1.2.5. □

We can now define an auxiliary predicate which we will use to prove soundness:

$$
\begin{aligned}
&\Gamma \models_n T \ \mathsf{type} \triangleq \\
&\quad \forall m \leq n.\ \Delta \vdash_m \gamma : \Gamma \circledR \rho \implies \Delta \vdash_m T[\gamma] \circledR [\![T]\!]_\rho\ \mathsf{type}_\omega \\
&\Gamma \models_n t : T \triangleq \\
&\quad \forall m \leq n.\ \Delta \vdash_m \gamma : \Gamma \circledR \rho \implies \Delta \vdash_m t[\gamma] : T[\gamma] \circledR [\![t]\!]_\rho \in_\omega [\![T]\!]_\rho \\
&\Gamma \models_n \delta : \Delta \triangleq \\
&\quad \forall m \leq n.\ \Gamma' \vdash_m \gamma : \Gamma \circledR \rho \implies \Gamma' \vdash_m \delta \circ \gamma : \Delta \circledR [\![\delta]\!]_\rho
\end{aligned}
$$

Theorem 4.4.7 (Soundness). *The following facts hold:*

1. *If $\Gamma \vdash T$ type then $\Gamma \models_n T$ type for any n.*
2. *If $\Gamma \vdash t : T$ then $\Gamma \models_n t : T$ for any n.*

3. *If* $\Gamma \vdash \delta : \Delta$ *then* $\Gamma \vDash_n \delta : \Delta$ *for any* n.

Proof. We prove these facts by mutual induction on the input derivation.

1. If $\Gamma \vdash T$ *type* then $\Gamma \vDash_n T$ type for any n.

   Case.

$$\frac{\Gamma\ \mathit{ctx}}{\Gamma \vdash \mathsf{U}_i\ \mathit{type}}$$

   In this case we have no induction hypothesis and we wish to show $\Gamma \vDash_n \mathsf{U}_i$ type for all n.

   In order to show this, suppose we have $m \leq n$, $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We must show $\Delta \vdash_m \mathsf{U}_i[\delta] \circledR \llbracket \mathsf{U}_i \rrbracket_\rho\ \text{type}_\omega$. First, we observe that $\llbracket \mathsf{U}_i \rrbracket_\rho = \mathbf{U}_i$ by definition independent of ρ.

   Therefore, in order to show $\Delta \vdash_m \mathsf{U}_i[\delta] \circledR \llbracket \mathsf{U}_i \rrbracket_\rho\ \text{type}_\omega$ we merely need to show that $i < \omega$ and $\Delta \vdash \mathsf{U}_i[\delta] = \mathsf{U}_i\ \mathit{type}$. Both are immediate.

   Case.

$$\frac{\Gamma\ \mathit{ctx}}{\Gamma \vdash \mathsf{nat}\ \mathit{type}}$$

   In this case we have no induction hypothesis and we wish to show $\Gamma \vDash_n \mathsf{nat}$ type for all n.

   In order to show this, suppose we have $m \leq n$, $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We must show $\Delta \vdash_m \mathsf{nat}[\delta] \circledR \llbracket \mathsf{nat} \rrbracket_\rho\ \text{type}_\omega$. First, we observe that $\llbracket \mathsf{nat} \rrbracket_\rho = \mathbf{nat}$.

   Therefore, in order to show $\Delta \vdash_m \mathsf{nat}[\delta] \circledR \mathbf{nat}\ \text{type}_\omega$ we merely need to show $\Delta \vdash \mathsf{nat}[\delta] = \mathsf{nat}\ \mathit{type}$. Both are immediate.

   Case.

$$\frac{\Gamma.\text{🔒} \vdash T\ \mathit{type}}{\Gamma \vdash \Box T\ \mathit{type}}$$

   For this, we have by induction hypothesis that $\Gamma.\text{🔒} \vDash_n T$ type for all n. We wish to show $\Gamma \vDash_n \Box T$ type for all n. Suppose we have some arbitrary n and suppose that we have some $m \leq n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We must show $\Delta \vdash_m (\Box T)[\delta] \circledR \llbracket \Box T \rrbracket_\rho\ \text{type}_\omega$.

   We have $\Delta\ \mathit{ctx}$ from Lemma 4.4.4. Therefore, $\Delta^{\text{🔓}} \vdash \mathsf{id} : \Delta$ from Lemma 1.2.5. Next, we use Lemma 4.4.3 with $\Delta \vdash_m \delta : \Gamma \circledR \rho$ to conclude that $\Delta^{\text{🔓}} \vdash_m \delta \circ \mathsf{id} : \Gamma \circledR \rho$. By Lemma 4.4.5 we then have $\Delta^{\text{🔓}} \vdash_m \delta : \Gamma \circledR \rho$. Finally, by definition we may conclude that $\Delta.\text{🔒} \vdash_{m'} \delta : \Gamma.\text{🔒} \circledR \rho$ for all m'.

   We may then instantiate our induction hypothesis with this fact to conclude that for all m' we have $\Delta.\text{🔒} \vdash_{m'} T[\delta] \circledR \llbracket T \rrbracket_\rho\ \text{type}_\omega$.

   Next, we have by definition that $\llbracket \Box T \rrbracket_\rho = \Box \llbracket T \rrbracket_\rho$. Again by definition we have that $\Delta \vdash_m (\Box T)[\delta] \circledR \Box \llbracket T \rrbracket_\rho\ \text{type}_\omega$ holds if and only if there is some T' such that $\Delta \vdash (\Box T)[\delta] = \Box T'\ \mathit{type}$ and such that for all m' we have $\Delta.\text{🔒} \vdash_{m'} T' \circledR \llbracket T \rrbracket_\rho\ \text{type}_\omega$. For this, we pick $T' = T[\delta]$. We have $\Delta \vdash (\Box T)[\delta] = \Box T'\ \mathit{type}$ and the next goal follows from our instantiated induction hypothesis.

   Case.

$$\frac{\Gamma \vdash T\ \mathit{type} \qquad \Gamma \vdash t_i : T}{\Gamma \vdash \mathsf{Id}(T, t_0, t_1)\ \mathit{type}}$$

   First, we have by induction hypothesis that $\Gamma \vDash_n T$ type and $\Gamma \vDash_n t_i : T$. We wish to show $\Gamma \vDash_n \mathsf{Id}(T, t_0, t_1)$ type.

   we suppose we have some $m \leq n$, $\Delta \vdash \delta : \Gamma$ such that $\Delta \vdash_m \delta : \Gamma \circledR \rho$, we wish to show $\Delta \vdash_m (\mathsf{Id}(T, t_0, t_1))[\delta] \circledR \llbracket \mathsf{Id}(T, t_0, t_1) \rrbracket_\rho\ \text{type}_\omega$.

First, we observe that we have $\Delta \vdash_m T[\delta] \circledR [\![T]\!]_\rho\ \mathsf{type}_\omega$, $\Delta \vdash_m t_0[\delta] : T[\delta] \circledR [\![t_0]\!]_\rho \in_\omega [\![T]\!]_\rho$, and $\Delta \vdash_m t_1[\delta] : T[\delta] \circledR [\![t_1]\!]_\rho \in_\omega [\![T]\!]_\rho$.

We observe next that in order to prove our goal that it suffices to show the following:

$$\Delta \vdash_m \mathsf{Id}(T[\delta], t_0[\delta], t_1[\delta]) \circledR \mathbf{Id}([\![T]\!]_\rho, [\![t_0]\!]_\rho, [\![t_1]\!]_\rho)\ \mathsf{type}_\omega$$

Therefore, we must show the following facts:

- $\Gamma \vdash \mathsf{Id}(T[\delta], t_0[\delta], t_1[\delta]) = \mathsf{Id}(T', t_0', t_1')\ \mathit{type}$ for some T', t_0', t_1';
- $\Gamma \vdash_n T' \circledR [\![T]\!]_\rho\ \mathsf{type}_\alpha$;
- $\Gamma \vdash_n t_i' : T' \circledR [\![t_i]\!]_\rho \in_\alpha [\![T]\!]_\rho$ for $i \in \{0, 1\}$.

The first of these follow by reflexivity and the remaining two follow our induction hypothesis.

Case.

$$\frac{\Gamma \vdash T_1\ \mathit{type} \qquad \Gamma.T_1 \vdash T_2\ \mathit{type}}{\Gamma \vdash \Pi(T_1, T_2)\ \mathit{type}}$$

First, we have by induction hypothesis that $\Gamma \vDash_n T_1$ type and $\Gamma.T_1 \vDash_n T_2$ type. We wish to show $\Gamma \vDash_n \Pi(T_1, T_2)$ type. Therefore, we suppose we have some $m \leq n$, $\Delta \vdash \delta : \Gamma$ such that $\Delta \vdash_m \delta : \Gamma \circledR \rho$, we wish to show $\Delta \vdash_m \Pi(T_1, T_2)[\delta] \circledR [\![\Pi(T_1, T_2)]\!]_\rho\ \mathsf{type}_\omega$.

First, we observe that the following holds:

$$\Delta \vdash \Pi(T_1, T_2)[\delta] = \Pi(T_1[\delta], T_2[(\delta \circ \mathsf{p}^1).\mathsf{var}_0])\ \mathit{type}$$

Therefore, by Lemma 4.3.6 it suffices to show $\Delta \vdash_m \Pi(T_1[\delta], T_2[(\delta \circ \mathsf{p}^1).\mathsf{var}_0]) \circledR [\![\Pi(T_1, T_2)]\!]_\rho\ \mathsf{type}_\omega$.
By calculation, we have $[\![\Pi(T_1, T_2)]\!]_\rho = \mathbf{\Pi}([\![T_1]\!]_\rho, T_2 \triangleleft \rho)$.

Now we may unfold this definition and see that we must show the following:

- $\Delta \vdash_m T_1' \circledR [\![T_1]\!]_\rho\ \mathsf{type}_\omega$
- if $m' \leq m$ and $r : \Delta' \leq \Delta$ such that $\Delta' \vdash_{m'} t : T_1'[r] \circledR a \in_\omega [\![T_1]\!]_\rho$ then $\Delta' \vdash_{m'} T_2'[r.t] \circledR [\![T_2]\!]_{\rho.a}\ \mathsf{type}_\omega$

For some T_i' such that $\Delta \vdash \Pi(T_1[\delta], T_2[(\delta \circ \mathsf{p}^1).\mathsf{var}_0]) = \Pi(T_1', T_2')\ \mathit{type}$. Now such a T_i' is straightforward.

Next, we have $\Delta \vdash_m T_1[\delta] \circledR [\![T_1]\!]_\rho\ \mathsf{type}_\omega$ from our induction hypothesis and the fact that $\Delta \vdash_m \delta : \Gamma \circledR \rho$.

Therefore, suppose we have some $m' \leq m$ and $r : \Delta' \leq \Delta$ along with $\Delta' \vdash_{m'} t : T_1[\delta][r] \circledR a \in_\omega [\![T_1]\!]_\rho$. We wish to show this:

$$\Delta' \vdash_{m'} T_2[(\delta \circ r).t] \circledR [\![T_2]\!]_{\rho.a}\ \mathsf{type}_\omega$$

In this, we have simplified the goal using the following fact:

$$\Delta' \vdash ((\delta \circ \mathsf{p}^1).\mathsf{var}_0) \circ (r.t) = (\delta \circ r).t : \Delta$$

In order to show this, we will use our induction hypothesis: $\Gamma.T_1 \vDash_n T_2$ type. It will suffice to show $\Delta' \vdash_{m'} (\delta \circ r).t : \Gamma.T_1 \circledR \rho.a$. In order to show this we must show $\Delta' \vdash_{m'} t : T_1[\delta \circ r] \circledR a \in_\omega [\![T_1]\!]_\rho$ and $\Delta' \vdash_{m'} \delta \circ r : \Gamma \circledR \rho$. The first follows from our assumption of $\Delta \vdash_m t : T_1[\delta] \circledR a \in_\omega [\![T_1]\!]_\rho$ and Lemmas 4.3.1 and 4.3.2. The second follows from $\Delta \vdash_m \delta : \Gamma \circledR \rho$ and Lemma 4.4.3.

Case.

$$\frac{\Gamma \vdash T_1\ \mathit{type} \qquad \Gamma.T_1 \vdash T_2\ \mathit{type}}{\Gamma \vdash \Sigma(T_1, T_2)\ \mathit{type}}$$

This case is identical to the previous case.

Case.

$$\frac{\Gamma \vdash T : \mathsf{U}_i}{\Gamma \vdash T \ \mathit{type}}$$

In this case we have $\Gamma \vDash_n T : \mathsf{U}_i$ and we wish to show $\Gamma \vDash_n T$ type. Suppose we have some $m \leq n$ and $\Delta \vdash \delta : \Gamma$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$, we wish to show $\Delta \vdash_m T[\delta] \circledR [\![T]\!]_\rho$ type_ω. We observe that from our induction hypothesis we then have the following:

$$\Delta \vdash_m T[\delta] : \mathsf{U}_i \circledR [\![T]\!]_\rho \in_\omega \mathbf{U}_i$$

By inversion then, we have $\Delta \vdash_m T[\delta] \circledR [\![T]\!]_\rho$ type_i. Since $i < \omega$ we have the desired conclusion from Lemma 4.3.8.

Case.

$$\frac{\Gamma \vdash \delta : \Delta \qquad \Delta \vdash T \ \mathit{type}}{\Gamma \vdash T[\delta] \ \mathit{type}}$$

In this case we have $\Delta \vDash_n T$ type and $\Gamma \vDash_n \delta : \Delta$ by induction hypothesis and wish to show $m \vdash \Gamma\ \mathit{type}T[\delta]$. Suppose we have some $m \leq n$ and $\Delta' \vdash \delta' : \Gamma$ and $\Delta' \vdash_m \delta' : \Gamma \circledR \rho$, we wish to show $\Delta' \vdash_m T[\delta \circ \delta'] \circledR [\![T]\!]_\rho$ type_ω.
First, we observe that $\Delta' \vdash \delta \circ \delta' : \Delta$. Furthermore, from $\Delta' \vDash_n \delta : \Gamma$ we have that $\Delta' \vdash_m \delta \circ \delta' : \Gamma \circledR [\![\delta]\!]_\rho$. We may then instantiate our other induction hypothesis with this to conclude that $\Delta' \vdash_m T[\delta \circ \delta'] \circledR [\![T]\!]_{[\![\delta]\!]_\rho}$ type_ω holds. By definition, we have $[\![T]\!]_{[\![\delta]\!]_\rho} = [\![T[\delta]]\!]_\rho$ concluding this case.

2. If $\Gamma \vdash t : T$ then $\Gamma \vDash_n t : T$ for any n.

Case.

$$\frac{\Gamma_1.T.\Gamma_2\ \mathit{ctx} \qquad \text{🔒} \notin \Gamma_2 \qquad k = \|\Gamma_2\|}{\Gamma_1.T.\Gamma_2 \vdash \mathsf{var}_k : T[\mathsf{p}^k]}$$

In this case we have no induction hypothesis. We wish to show $\Gamma_1.T.\Gamma_2 \vDash_n \mathsf{var}_k : T[\mathsf{p}^k]$. Suppose we have $m \leq n$, $\Delta \vdash \delta : \Gamma_1.T.\Gamma_2$, and $\Delta \vdash_m \delta : \Gamma_1.T.\Gamma_2 \circledR \rho$. We wish to show the following:

$$\Delta \vdash_m \mathsf{var}_k[\delta] : T[\mathsf{p}^k \circ \delta] \circledR [\![\mathsf{var}_k]\!]_\rho \in_\omega [\![T[\mathsf{p}^k]]\!]_\rho$$

We observe that since $\text{🔒} \notin \Gamma_2$ we have by inversion on $\Delta \vdash_m \delta : \Gamma_1.T.\Gamma_2 \circledR \rho$ that $\rho = \rho'.v_1.\dots.v_k$ and $\Delta \vdash \delta = \delta'.t_1.\dots.t_k : \Gamma_1.T.\Gamma_2$ such that $\Delta \vdash_m \delta' : \Gamma_1 \circledR \rho'$ and $\Delta \vdash_m t_1 : T[\delta'] \circledR v_1 \in_\omega [\![T]\!]_{\rho'}$.
Next we observe that $[\![\mathsf{var}_k]\!]_\rho = \rho(k) = v_1$ and $\Delta \vdash \mathsf{var}_k[\delta] = t_1 : T[\delta']$. We note that $\Delta \vdash \mathsf{p}^k \circ \delta = \delta' : \Gamma_1$ and so we may turn the latter fact into $\Delta \vdash \mathsf{var}_k[\delta] = t_1 : T[\mathsf{p}^k \circ \delta]$.
From this equality of substitutions we also have $\Delta \vdash_m t_1 : T[\mathsf{p}^k \circ \delta] \circledR v_1 \in_\omega [\![T]\!]_{\rho'}$ by Lemma 4.3.6. By calculation we also have that $[\![T]\!]_{\rho'} = [\![T[\mathsf{p}^k]]\!]_\rho$ and so we have $\Delta \vdash_m t_1 : T[\mathsf{p}^k \circ \delta] \circledR v_1 \in_\omega [\![T[\mathsf{p}^k]]\!]_\rho$.
Finally, we are done by Lemma 4.3.7 and $\Delta \vdash \mathsf{var}_k[\delta] = t_1 : T[\mathsf{p}^k \circ \delta]$.

Case.

$$\frac{\Gamma \vdash T_0\ \mathit{type} \qquad \Gamma.T_0 \vdash t : T_1}{\Gamma \vdash \lambda(t) : \Pi(T_0, T_1)}$$

In this case, we have $\Gamma \vDash_n T_0$ type and $\Gamma.T_0 \vDash_n t : T_1$ by induction hypothesis. We wish to show $\Gamma \vDash_n \lambda(t) : \Pi(T_0, T_1)$.
Suppose we have some $m \leq n$, $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We must show the following:

$$\Delta \vdash_m \lambda(t)[\delta] : (\Pi(T_0, T_1))[\delta] \circledR [\![\lambda t]\!]_\rho \in_\omega [\![\Pi(T_0, T_1)]\!]_\rho$$

First, we observe by calculation that $[\![\lambda(t)]\!]_\rho = \lambda(t \triangleleft \rho)$ and $[\![\Pi(T_0, T_1)]\!]_\rho = \mathbf{\Pi}([\![T_0]\!]_\rho, T_1 \triangleleft \rho)$. Next, we will use the following two definitional equalities.

$$\Delta \vdash \Pi(T_0, T_1)[\delta] = \Pi(T_0[\delta], T_1[\delta \circ \mathsf{p}^1.\mathsf{var}_0])\ \mathit{type}$$
$$\Delta \vdash (\lambda(t))[\delta] = \lambda(t[\delta \circ \mathsf{p}^1.\mathsf{var}_0]) : \Pi(T_0, T_1)[\delta]$$

We may then simplify our goal by Lemmas 4.3.6 and 4.3.7 to the following:

$$\Delta \vdash_m \lambda(t[\delta \circ \mathsf{p}^1.\mathsf{var}_0]) : \Pi(T_0[\delta], T_1[\delta \circ \mathsf{p}^1.\mathsf{var}_0]) \circledR \lambda(t \triangleleft \rho) \in_\omega \mathbf{\Pi}([\![T_0]\!]_\rho, T_1 \triangleleft \rho)$$

In order to show this, we unfold the definition. It suffices to show that two facts hold:

Subgoal.

$$\Delta \vdash_m T_0[\delta] \circledR [\![T_0]\!]_\rho\ \mathsf{type}_\omega$$

This follows from our induction hypothesis. We instantiate $\Gamma \vDash_n T_0$ type with $m \leq n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$ and the conclusion is immediate.

Subgoal.

For all $m' \leq m$ and $r : \Delta' \leq \Delta$ if $\Delta' \vdash_{m'} t' : T_0[\delta \circ r] \circledR v \in_\omega [\![T_0]\!]_\rho$ then we have the following:

$$\Delta' \vdash_{m'} (\lambda(t[\delta \circ \mathsf{p}^1.\mathsf{var}_0]))[r](t') : T_1[\delta \circ \mathsf{p}^1.\mathsf{var}_0][r.t'] \circledR \underline{\mathbf{app}}(\lambda(t \triangleleft \rho), v) \in_\omega T_1 \triangleleft \rho[v]$$

First, we use Lemmas 4.3.6 and 4.3.7 again to simplify our goal to the following:

$$\Delta' \vdash_{m'} t[(\delta \circ r).t'] : T_1[(\delta \circ r).t'] \circledR [\![t]\!]_{\rho.v} \in_\omega [\![T_1]\!]_{\rho.v}$$

In order to show this we will use our second induction hypothesis. We pick $m' \leq n$ by transitivity. If we can show that $\Delta' \vdash_{m'} (\delta \circ r).t' : \Gamma.T_0 \circledR \rho.v$ we are done. We observe from the definition that since $\Delta' \vdash_{m'} t' : T_0[\delta \circ r] \circledR v \in_\omega [\![T_0]\!]_\rho$ holds by assumption we merely need to show $\Delta' \vdash_{m'} \delta \circ r : \Gamma \circledR \rho$. Next, by Lemma 4.4.3 it suffices to show $\Delta \vdash_m \delta : \Gamma \circledR \rho$ but this is immediate by assumption.

Case.

$$\frac{\Gamma \vdash T_0\ \mathit{type} \qquad \Gamma.T_0 \vdash T_1\ \mathit{type} \qquad \Gamma \vdash t_0 : \Pi(T_0, T_1) \qquad \Gamma \vdash t_1 : T_0}{\Gamma \vdash t_0(t_1) : T_1[\mathsf{id}.t_1]}$$

We have by induction hypothesis that $\Gamma \vDash_n T_0$ type, $\Gamma.T_0 \vDash_n T_1$ type, $\Gamma \vDash_n t_0 : \Pi(T_0, T_1)$ and $\Gamma \vDash_n t_1 : T_0$. We wish to show $\Gamma \vDash_n t_0(t_1) : T_1[\mathsf{id}.t_0]$. We set $T = \Pi(T_0, T_1)$.

Suppose we have some $m \leq n$, $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We must show the following:

$$\Delta \vdash_m t_0(t_1)[\delta] : T_1[(\mathsf{id}.t_1) \circ \delta] \circledR \underline{\mathbf{app}}([\![t_0]\!]_\rho, [\![t_1]\!]_\rho) \in_\omega [\![T_1]\!]_{\rho.[\![t_1]\!]_\rho}$$

We instantiate our induction hypotheses with m, δ, and ρ. We then have $\Delta \vdash_m t_0 : T[\delta] \circledR [\![t_0]\!]_\rho \in_\omega [\![T]\!]_\rho$ and $\Delta \vdash_m t_1 : T_0[\delta] \circledR [\![t_1]\!]_\rho \in_\omega [\![T_0]\!]_\rho$.

By inversion on the first of these facts we must then have that there is some T_0' and T_1' such that $\Delta \vdash_m T_0' \circledR [\![T_0]\!]_\rho\ \mathsf{type}_\omega$ and such that for all $\Delta \vdash_m t' : T_0' \circledR v \in_\omega [\![T_0]\!]_\rho$ we have $\Delta \vdash_m t_0(t') : T_1'[\mathsf{id}.t'] \circledR \underline{\mathbf{app}}([\![t_0]\!]_\rho, v) \in_\omega [\![T_1]\!]_{\rho.v}$

Now, we observe that by Corollary 4.3.12 we must have $\Delta \vdash T_0[\delta] = T_0'$ *type*. Therefore, from our second induction hypothesis and the second fact we have obtained from inversion, we may conclude the following:

$$\Delta \vdash_m t_0(t_1) : T_1'[\mathsf{id}.t_1] \circledR [\![t_0(t_1)]\!]_\rho \in_\omega [\![T_1]\!]_{\rho.[\![t_1]\!]_\rho}$$

In order to obtain the desired conclusion, therefore, we must show that $\Delta \vdash T_1[\delta.t_1] = T_1'[\mathsf{id}.t_1]$ *type* holds. This follows form Corollary 4.3.12 and our induction hypothesis of $\Gamma.T_0 \vDash_n T_1$ type. From the latter we have $\Delta \vdash_m T_1[\mathsf{id}.t_1] \circledR [\![T_1]\!]_{\rho.[\![t_1]\!]_\rho}$ type_ω. From our earlier conclusion and Lemma 4.3.9 we may have $\Delta \vdash_m T_1'[\mathsf{id}.t_1] \circledR [\![T_1]\!]_{\rho.[\![t_1]\!]_\rho}$ type_ω. Therefore, we have the desired equality of types by Corollary 4.3.12.

Case.

$$\frac{\Gamma \vdash A : \mathsf{U}_i \qquad \Gamma.A \vdash B : \mathsf{U}_i}{\Gamma \vdash \Pi(A, B) : \mathsf{U}_i}$$

Identical to the case for $\Gamma \vdash \Pi(A, B)$ *type*.

Case.

$$\frac{\Gamma \vdash t_0 : T_0 \qquad \Gamma.T_0 \vdash T_1 \ \mathit{type} \qquad \Gamma \vdash t_1 : T_1[\mathsf{id}.t_0]}{\Gamma \vdash \langle t_0, t_1 \rangle : \Sigma(T_0, T_1)}$$

In this case, by induction hypothesis we have $\Gamma \vDash_n t_0 : T_0$, $\Gamma.T_0 \vDash_n T_1$ type, and $\Gamma \vDash_n t_1 : T_1[\mathsf{id}.t_0]$. We wish to show $\Gamma \vDash_n \langle t_0, t_1 \rangle : \Sigma(T_0, T_1)$.

Suppose we have some $m \leq n$, $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We must show the following:

$$\Delta \vdash_n \langle t_0, t_1 \rangle[\delta] : (\Sigma(T_0, T_1))[\delta] \circledR [\![\langle t_0, t_1 \rangle]\!]_\rho \in_\omega [\![\Sigma(T_0, T_1)]\!]_\rho$$

First, we observe that $[\![\Sigma(T_0, T_1)]\!]_\rho = \Sigma([\![T_0]\!]_\rho, [\![T_1]\!]_\rho)$. Therefore, we must show that that $\Delta \vdash (\Sigma(T_0, T_1))[\delta] = \Sigma(T_0', T_1')$ *type*, $\Delta \vdash \langle t_0, t_1 \rangle[\delta] : \Sigma(T_0', T_1')$, and the following three facts:

a) $\forall m' \leq m, r : \Delta' \leq \Delta.\ \Delta' \vdash_{m'} t' : T_0'[r] \circledR a \in_\omega [\![T_0]\!]_\rho \implies \Delta' \vdash_{m'} T_1'[r.t'] \circledR T_1 \triangleleft \rho[a]$ type_ω

b) $\Delta \vdash_m t_0[\delta] : T_0' \circledR \underline{\mathbf{fst}}([\![\langle t_0, t_1 \rangle]\!]_\rho) \in_\omega [\![T_0]\!]_\rho$

c) $\Delta \vdash_m t_1[\delta] : T_1'[\mathsf{id}.t_0[\delta]] \circledR \underline{\mathbf{snd}}([\![\langle t_0, t_1 \rangle]\!]_\rho) \in_\omega T_1 \triangleleft \rho[\underline{\mathbf{fst}}([\![\langle t_0, t_1 \rangle]\!]_\rho)]$

We have simplified these goals without further comment by Lemma 4.3.7 to save space.

We choose $T_0' = T_0[\delta]$ and $T_1' = T_1[(\delta \circ \mathsf{p}^1).\mathsf{var}_0]$. This immediately gives us $\Delta \vdash \langle t_0, t_1 \rangle[\delta] : \Sigma(T_0', T_1')$ so we merely need to show the above three facts.

The first fact then follows from our induction hypothesis of $\Gamma.T_0 \vDash_n T_1$ type. For the second, we observe by that $\underline{\mathbf{fst}}([\![\langle t_0, t_1 \rangle]\!]_\rho) = [\![t_0]\!]_\rho$ and so this goal is precisely our induction hypothesis of $\Gamma \vDash_n t_0 : T_0$. For the third, we observe that $\underline{\mathbf{snd}}([\![\langle t_0, t_1 \rangle]\!]_\rho) = [\![t_1]\!]_\rho$. This simplifies our goal to the following (again using Lemma 4.3.7):

$$\Delta \vdash_m t_1[\delta] : T_1[\delta.t_0[\delta]] \circledR [\![t_1]\!]_\rho \in_\omega [\![T_1]\!]_{\rho.[\![t_0]\!]_\rho}$$

This is again handled by our induction hypothesis.

Case.

$$\frac{\Gamma \vdash T_0 \ \mathit{type} \qquad \Gamma \vdash t : \Sigma(T_0, T_1)}{\Gamma \vdash \mathsf{fst}(t) : T_0}$$

In this case we have by induction hypothesis that $\Gamma \vDash_n T_0$ type and $\Gamma \vDash_n t : \Sigma(T_0, T_1)$. We wish to show $\Gamma \vDash_n \mathsf{fst}(t) : T_0$.

Suppose we have $m \leq n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We wish to show the following:

$$\Delta \vdash_m (\mathsf{fst}(t))[\delta] : T_0[\delta] \circledR \underline{\mathbf{fst}}([\![t]\!]_\rho) \in_\omega [\![T_0]\!]_\rho$$

We start by instantiating our induction hypothesis of $\Gamma \vDash_n t : \Sigma(T_0, T_1)$. This tells us that the following holds:

$$\Delta \vdash_m t[\delta] : \Sigma(T_0, T_1)[\delta] \circledR [\![t]\!]_\rho \in_\omega [\![\Sigma(T_0, T_1)]\!]_\rho$$

Therefore, we have $\Delta \vdash \Sigma(T_0, T_1)[\delta] = \Sigma(T_0', T_1')$ *type* such that, in particular, $\Delta \vdash_m \mathsf{fst}(t[\delta]) : T_0' \circledR \underline{\mathbf{fst}}(\llbracket t \rrbracket_\rho) \in_\omega \llbracket T_0 \rrbracket_\rho$. Now we may use Corollary 4.3.12 with $\Gamma \vDash_n T_0$ type to conclude that $\Delta \vdash T_0[\delta] = T_0'$ *type*. Finally, by Lemmas 4.3.6 and 4.3.7 we then have the desired goal:

$$\Delta \vdash_m (\mathsf{fst}(t))[\delta] : T_0[\delta] \circledR \underline{\mathbf{fst}}(\llbracket t \rrbracket_\rho) \in_\omega \llbracket T_0 \rrbracket_\rho$$

Case.

$$\frac{\Gamma \vdash T_0 \ \textit{type} \qquad \Gamma.T_0 \vdash T_1 \ \textit{type} \qquad \Gamma \vdash t : \Sigma(T_0, T_1)}{\Gamma \vdash \mathsf{snd}(t) : T_1[\mathsf{id}.(\mathsf{fst}(t))]}$$

In this case we have by induction hypothesis that $\Gamma \vDash_n T_0$ type, $\Gamma.T_0 \vDash_n T_1$ type and $\Gamma \vDash_n t : \Sigma(T_0, T_1)$. We wish to show $\Gamma \vDash_n \mathsf{fst}(t) : T_0$.
Suppose we have $m \leq n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We wish to show the following:

$$\Delta \vdash_m (\mathsf{snd}(t))[\delta] : T_1[\delta.\mathsf{fst}(t[\delta])] \circledR \underline{\mathbf{snd}}(\llbracket t \rrbracket_\rho) \in_\omega \llbracket T_1 \rrbracket_{\rho.\llbracket \mathsf{fst}(t) \rrbracket_\rho}$$

We start by instantiating our induction hypothesis of $\Gamma \vDash_n t : \Sigma(T_0, T_1)$. This tells us that the following holds:

$$\Delta \vdash_m t[\delta] : (\Sigma(T_0, T_1))[\delta] \circledR \llbracket t \rrbracket_\rho \in_\omega \llbracket \Sigma(T_0, T_1) \rrbracket_\rho$$

Inversion on this tells us that there is some $\Delta \vdash (\Sigma(T_0, T_1))[\delta] = \Sigma(T_0', T_1')$ *type* such that the following holds:

$$\Delta \vdash_m \mathsf{fst}((t[\delta])) : T_0' \circledR \underline{\mathbf{fst}}(\llbracket t \rrbracket_\rho) \in_\omega \llbracket T_0 \rrbracket_\rho$$
$$\Delta \vdash_m \mathsf{snd}((t[\delta])) : T_1'[\mathsf{id}.\mathsf{fst}((t[\delta]))] \circledR \underline{\mathbf{snd}}(\llbracket t \rrbracket_\rho) \in_\omega \llbracket T_1 \rrbracket_{\rho.\underline{\mathbf{fst}}(\llbracket t \rrbracket_\rho)}$$

From the first fact, Corollary 4.3.12 and our induction hypothesis that $\Gamma \vDash_n T_0$ type we may conclude that $\Delta \vdash T_0[\delta] = T_0'$ *type* holds. We then have from the second fact, Corollary 4.3.12, and our induction hypothesis that $\Gamma.T_0 \vDash_n T_1$ type that the following equality is true:

$$\Delta \vdash T_1[\delta.(\mathsf{fst}(t[\delta]))] = T_1'[\mathsf{id}.(\mathsf{fst}(t[\delta]))] \ \textit{type}$$

Therefore, we may conclude from Lemmas 4.3.6 and 4.3.7 that our desired goal holds.

Case.

$$\frac{\Gamma \vdash A : \mathsf{U}_i \qquad \Gamma.A \vdash B : \mathsf{U}_i}{\Gamma \vdash \Sigma(A, B) : \mathsf{U}_i}$$

Identical to the case for $\Gamma \vdash \Sigma(A, B)$ *type*.

Case.

$$\frac{\Gamma \ \textit{ctx}}{\Gamma \vdash \mathsf{zero} : \mathsf{nat}}$$

In this case we wish to show that $\Gamma \vDash_n \mathsf{zero} : \mathsf{nat}$ holds. Suppose that we have $m \leq n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We must show that $\Delta \vdash_m \mathsf{zero}[\delta] : \mathsf{nat}[\delta] \circledR \mathbf{zero} \in_\omega \mathbf{nat}$. In order to show this it suffices to show $\Delta \vdash_m \mathsf{zero} : \mathsf{nat} \circledR \mathbf{zero} \in_\omega \mathbf{nat}$ and this is immediate by definition.

Case.

$$\frac{\Gamma \vdash t : \mathsf{nat}}{\Gamma \vdash \mathsf{succ}(t) : \mathsf{nat}}$$

In this case we wish to show that $\Gamma \vDash_n \mathsf{succ}(t) : \mathsf{nat}$ holds and we have by induction hypothesis that $\Gamma \vDash_n t : \mathsf{nat}$. Suppose that we have $m \leq n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We must show $\Delta \vdash_m \mathsf{succ}(t)[\delta] : \mathsf{nat}[\delta] \circledR \mathbf{succ}(\llbracket t \rrbracket_\rho) \in_\omega \mathbf{nat}$.
First, observe by our induction hypothesis that we have $\Delta \vdash_m t[\delta] : \mathsf{nat} \circledR \llbracket t \rrbracket_\rho \in_\omega \mathbf{nat}$. Therefore, the goal follows by definition.

Case.

$$\frac{\Gamma.\mathsf{nat} \vdash T\ \mathit{type} \qquad \Gamma \vdash t_0 : \mathsf{nat} \qquad \Gamma \vdash t_1 : T[\mathsf{id}.\mathsf{zero}] \qquad \Gamma.\mathsf{nat}.T \vdash t_2 : T[\mathsf{p}^2.\mathsf{succ}(\mathsf{var}_1)]}{\Gamma \vdash \mathsf{natrec}(T, t_0, t_1, t_2) : T[\mathsf{id}.t_1]}$$

In this case we have by induction hypothesis that $\Gamma.\mathsf{nat} \vDash_n T$ type, $\Gamma \vDash_n t_0 : \mathsf{nat}$, $\Gamma \vDash_n t_1 : T[\mathsf{id}.\mathsf{zero}]$, and $\Gamma.\mathsf{nat}.T \vDash_n t_2 : T[\mathsf{p}^2.\mathsf{succ}(\mathsf{var}_1)]$. We wish to show that $\Gamma \vDash_n \mathsf{natrec}(T, t_0, t_1, t_2) : T[\mathsf{id}.t_0]$ holds.

For this, suppose we have some $m \leq n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We first observe that we have $\Delta \vdash_m t_0[\delta] : \mathsf{nat} \circledR [\![t_0]\!]_\rho \in_\omega \mathbf{nat}$. This relation is inductively defined so we proceed by induction. There are 3 subcases to consider:

Subcase. $\Delta \vdash t_0[\delta] = \mathsf{zero} : \mathsf{nat}$ and $[\![t_0]\!]_\rho = \mathsf{zero}$.

In this case we wish to show that the following holds:

$$\Delta \vdash_m \mathsf{natrec}(T, t_0, t_1, t_2)[\delta] : T[\mathsf{id}.t_0][\delta] \circledR [\![\mathsf{natrec}(T, t_0, t_1, t_2)]\!]_\rho \in_\omega [\![T[\mathsf{id}.t_0]]\!]_\rho$$

We can reduce this as $\mathsf{natrec}(-,-,-,-)$ reduces at zero. It suffices to show the following instead:

$$\Delta \vdash_m t_1[\delta] : T[\mathsf{id}.\mathsf{zero}][\delta] \circledR [\![t_1]\!]_\rho \in_\omega [\![T[\mathsf{id}.\mathsf{zero}]]\!]_\rho$$

However, this follows precisely from our induction hypothesis that $\Gamma \vDash_n t_1 : T[\mathsf{id}.\mathsf{zero}]$.

Subcase. $\Delta \vdash t_0[\delta] = \mathsf{succ}(t_0') : \mathsf{nat}$, $[\![t_0]\!]_\rho = \mathbf{succ}(v)$ and $\Delta \vdash_m t_0' : \mathsf{nat} \circledR v \in_\omega \rho$.

In this case we wish to show that the following holds (after some simplifications):

$$\Delta \vdash_m t_2[\delta.t_0[\delta].\mathsf{rec}(\ldots)] : T[\delta.\mathsf{succ}(t_0')] \circledR [\![t_2]\!]_{\rho.v.\underline{\mathbf{natrec}}(T\triangleleft\rho, [\![t_1]\!]_\rho, t_2\triangleleft\rho, v)} \in_\omega [\![T[\mathsf{id}.t_0]]\!]_\rho$$

We have by induction hypothesis that the following holds:

$$\Delta \vdash_m \mathsf{rec}(\ldots) : T[\delta.t_0'] \circledR \underline{\mathbf{natrec}}(T\triangleleft\rho, [\![t_1]\!]_\rho, t_2\triangleleft\rho, v) \in_\omega [\![T]\!]_{\rho.v}$$

Therefore, the goal holds from our induction hypothesis of $\Gamma.\mathsf{nat}.T \vDash_n t_2 : T[\mathsf{p}^2.\mathsf{succ}(\mathsf{var}_1)]$.

Subcase. We have $[\![t_0]\!]_\rho = \uparrow^{\mathbf{nat}} e$ and for all $r : \Delta' \leq \Delta$ we have $\lceil e \rceil_{\|\Delta'\|} = t'$ and $\Delta' \vdash t_0[r \circ \delta] = t' : \mathsf{nat}$.

In this case we wish to show

$$\Delta \vdash_m \mathsf{natrec}(T, t_0, t_1, t_2)[\delta] : T[\delta.t_0[\delta]] \circledR e.\mathbf{natrec}(T\triangleleft\rho, [\![t_1]\!]_\rho, t_2\triangleleft\rho) \in_\omega [\![T]\!]_{\rho.\uparrow^{\mathbf{nat}} e}$$

In this case we use Lemma 4.3.11. Specifically, we must show that for all $r : \Delta' \leq \Delta$ that $\lceil e.\mathbf{natrec}(T\triangleleft\rho, [\![t_1]\!]_\rho, t_2\triangleleft\rho) \rceil_{\|\Delta'\|} = t'$ such that the following holds:

$$\Delta' \vdash \mathsf{natrec}(T, t_0, t_1, t_2)[r \circ \delta] = t' : T[r \circ \delta.t_0[\delta]]$$

This follows from our assumption about e as well as our induction hypothesis of $\Gamma \vDash_n t_0 : \mathsf{nat}$, $\Gamma \vDash_n t_1 : T[\mathsf{id}.\mathsf{zero}]$, and $\Gamma.\mathsf{nat}.T \vDash_n t_2 : T[\mathsf{p}^2.\mathsf{succ}(\mathsf{var}_1)]$.

Case.

$$\frac{\Gamma\ \mathit{ctx}}{\Gamma \vdash \mathsf{nat} : \mathsf{U}_i}$$

Identical to the case for $\Gamma \vdash \mathsf{nat}$ *type*.

$$\frac{\Gamma \vdash T : \mathsf{U}_i \qquad \Gamma \vdash t_i : T}{\Gamma \vdash \mathsf{Id}(T, t_0, t_1) : \mathsf{U}_i}$$

Identical to the case for $\Gamma \vdash \mathsf{Id}(T, t_0, t_1)$ *type*.

Case.

$$\frac{\Gamma \vdash T\ \mathit{type} \qquad \Gamma \vdash t : T}{\Gamma \vdash \mathsf{refl}(t) : \mathsf{Id}(T, t, t)}$$

Suppose that $\Gamma \vDash_n T$ type and $\Gamma \vDash_n t : T$, we wish to show $\Gamma \vDash_n \mathsf{refl}(t) : \mathsf{Id}(T, t, t)$.
For this, suppose we have $m \le n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We wish to show the following:

$$\Delta \vdash_m \mathsf{refl}(t)[\delta] : (\mathsf{Id}(T, t, t))[\delta] \circledR [\![\mathsf{refl}(t)]\!]_\rho \in_\omega [\![\mathsf{Id}(T, t, t)]\!]_\rho$$

We first observe that we can simplify this goal to the following:

$$\Delta \vdash_m \mathsf{refl}(t[\delta]) : \mathsf{Id}(T[\delta], t[\delta], t[\delta]) \circledR \mathbf{refl}([\![t]\!]_\rho) \in_\omega \mathbf{Id}([\![T]\!]_\rho, [\![t]\!]_\rho, [\![t]\!]_\rho)$$

By unfolding the definition of the logical relation at $\mathbf{Id}([\![T]\!]_\rho, [\![t]\!]_\rho, [\![t]\!]_\rho)$, we must show the following:

- $\Delta \vdash_n T[\delta] \circledR [\![T]\!]_\rho\ \mathsf{type}_\alpha$
- $\Delta \vdash_n t[\delta] : T[\delta] \circledR [\![t]\!]_\rho \in_\alpha [\![T]\!]_\rho$

Both of these follow from our induction hypothesis.

Case.

$$\frac{\Gamma \vdash T\ \mathit{type} \qquad \Gamma \vdash u_1, u_2 : T \qquad \Gamma.T.T[\mathsf{p}^1].\mathsf{Id}(T[\mathsf{p}^2], \mathsf{var}_1, \mathsf{var}_0) \vdash C\ \mathit{type} \\ \Gamma.T \vdash t_1 : C[\mathsf{id}.\mathsf{var}_0.\mathsf{var}_0.\mathsf{refl}(\mathsf{var}_0)] \qquad \Gamma \vdash t_2 : \mathsf{Id}(T, u_1, u_2)}{\Gamma \vdash \mathsf{J}(C, t_1, t_2) : C[\mathsf{id}.u_1.u_2.t_2]}$$

In this case we have from our induction hypothesis that $\Gamma \vDash_n T$ type, $\Gamma \vDash_n u_1, u_2 : T$, $\Gamma.T.T[\mathsf{p}^1].\mathsf{Id}(T[\mathsf{p}^2], \mathsf{var}_1, \mathsf{var}_0) \vDash_n C$ type, $\Gamma.T \vDash_n t_1 : C[\mathsf{id}.\mathsf{var}_0.\mathsf{var}_0.\mathsf{refl}(\mathsf{var}_0)]$, and $\Gamma \vDash_n t_2 : \mathsf{Id}(T, u_1, u_2)$.
We wish to show $\Gamma \vDash_n \mathsf{J}(C, t_1, t_2) : C[\mathsf{id}.u_1.u_2.t_2]$.
First, assume that we have $m \le n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We wish to show the following:

$$\Delta \vdash_m \mathsf{J}(C, t_1, t_2)[\delta] : C[\delta.u_1[\delta].u_2[\delta].t_2[\delta]] \circledR [\![\mathsf{J}(C, t_1, t_2)]\!]_\rho \in_\omega [\![C]\!]_{\rho.[\![u_1]\!]_\rho.[\![u_2]\!]_\rho.[\![t_2]\!]_\rho}$$

In order to show this, we observe that by induction hypothesis we have $\Delta \vdash_m t_2[\delta] : \mathsf{Id}(T, u_1, u_2)[\delta] \circledR [\![t_2]\!]_\rho \in_\omega [\![\mathsf{Id}(T, u_1, u_2)]\!]_\rho$. By inversion on this fact we have that one of the following two cases applies:

- $[\![t_2]\!]_\rho = \uparrow^{-} e$ and when $r : \Delta' \le \Delta$, then $\lceil e \rceil_{\|\Delta'\|} = t'$ such that $\Delta' \vdash t_2[\delta][r] = t' : T[\delta][r]$.
- $\Delta \vdash t_2[\delta] = \mathsf{refl}(t') : \mathsf{Id}(T, u_1, u_2)[\delta]$ and $[\![t_2]\!]_\rho = \mathbf{refl}(v')$ for some t', v' such that $\Delta \vdash t' = u_i[\delta] : T[\delta]$.

We proceed by cases on this. In the first case we have that $[\![t_2]\!]_\rho = \uparrow^{-} e$. We also observe from our induction hypothesis that the following equality holds:

$$[\![\mathsf{J}(C, t_1, t_2)]\!]_\rho = \uparrow^{[\![C]\!]_{\rho.[\![u_1]\!]_\rho.[\![u_2]\!]_\rho.[\![t_2]\!]_\rho}} e.\mathbf{J}(C \triangleleft \rho, t_1 \triangleleft \rho, [\![T]\!]_\rho, [\![u_1]\!]_\rho, [\![u_2]\!]_\rho)$$

In order to show our goal then, it suffices to show that for all $r : \Delta' \le \Delta$ that there is some t' such that

$$\lceil e.\mathbf{J}(C \triangleleft \rho, t_1 \triangleleft \rho, [\![T]\!]_\rho, [\![u_1]\!]_\rho, [\![u_2]\!]_\rho) \rceil_{\|\Delta'\|} = t'$$

Moreover, we must have the following equality:

$$\Delta' \vdash \mathsf{J}(C, t_1, t_2)[r \circ \delta] = t' : C[\mathsf{id}.u_1.u_2.t_2][r \circ \delta]$$

However, this holds using our induction hypothesis and the assumption that for all $r : \Delta' \le \Delta$, then $\lceil e \rceil_{\|\Delta'\|} = t''$ such that $\Delta' \vdash t_2[\delta][r] = t'' : T[\delta][r]$

For the second case, we have that $[\![t_2]\!]_\rho = \mathbf{refl}(v')$ and $\Delta \vdash t_2[\delta] = \mathsf{refl}(t') : \mathsf{Id}(T, u_1, u_2)[\delta]$. In this case, we may simplify our goal to the following:

$$\Delta \vdash_m t_1[\delta.t'] : C[\delta.u_1[\delta].u_2[\delta].t_2[\delta]] \circledR [\![t_1]\!]_{\rho.v'} \in_\omega [\![C]\!]_{\rho.[\![u_1]\!]_\rho.[\![u_2]\!]_\rho.[\![t_2]\!]_\rho}$$

In this case we wish to apply our induction hypothesis for t_1:

$$\Gamma.T \vDash_n t_1 : C[\mathsf{id}.\mathsf{var}_0.\mathsf{var}_0.\mathsf{refl}(\mathsf{var}_0)]$$

This allows us to conclude the following:

$$\Delta \vdash_m t_1[\delta.t'] : C[\delta.t'.t'.\mathsf{refl}(t')] \circledR [\![t_1]\!]_{\rho.v'} \in_\omega [\![C]\!]_{\rho.v'.v'.\mathbf{refl}(v')}$$

Now, we may use Lemma 4.3.6 to simplify this to the following:

$$\Delta \vdash_m t_1[\delta.t'] : C[\delta.u_1[\delta].u_2[\delta].t_2[\delta]] \circledR [\![t_1]\!]_{\rho.v'} \in_\omega [\![C]\!]_{\rho.v'.v'.\mathbf{refl}(v')}$$

Finally, we have $\Gamma.T.T[\mathsf{p}^1].\mathsf{Id}(T[\mathsf{p}^2], \mathsf{var}_1, \mathsf{var}_0) \vdash C$ *type*. We use Theorem 3.3.5 together with the following pair of environments:

$$m \Vdash \rho.v'.v'.\mathbf{refl}(v') = \rho.[\![u_1]\!]_\rho.[\![u_2]\!]_\rho.[\![t_2]\!]_\rho : \Gamma.T.T[\mathsf{p}^1].\mathsf{Id}(T[\mathsf{p}^2], \mathsf{var}_1, \mathsf{var}_0)$$

This tells us that $\tau_\omega \models_m [\![C]\!]_{\rho.v'.v'.\mathbf{refl}(v')} \sim [\![C]\!]_{\rho.[\![u_1]\!]_\rho.[\![u_2]\!]_\rho.[\![t_2]\!]_\rho}$. Our goal then follows from Lemma 4.3.5.

Case.

$$\frac{\Gamma.\text{🔒} \vdash t : T}{\Gamma \vdash [t]_{\text{🔒}} : \Box T}$$

We have by induction hypothesis in this case that $\Gamma.\text{🔒} \vDash_n t : T$. We wish to show $\Gamma \vDash_n [t]_{\text{🔒}} : \Box T$. For this, suppose we have $m \leq n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We wish to show the following:

$$\Delta \vdash_m [t]_{\text{🔒}}[\delta] : (\Box T)[\delta] \circledR [\![[t]_{\text{🔒}}]\!]_\rho \in_\omega [\![\Box T]\!]_\rho$$

We can calculate to reduce this to the following:

$$\Delta \vdash_m [t[\delta]]_{\text{🔒}} : \Box T[\delta] \circledR \mathbf{shut}([\![t]\!]_\rho) \in_\omega \Box [\![T]\!]_\rho$$

Now in order to show this it suffices to show for all m',

$$\Delta.\text{🔒} \vdash_{m'} [[t[\delta]]_{\text{🔒}}]_{\text{🔓}} : T[\delta] \circledR \underline{\mathbf{open}}(\mathbf{shut}([\![t]\!]_\rho)) \in_\omega [\![T]\!]_\rho$$

By calculation this simplifies to the following $\Delta.\text{🔒} \vdash_{m'} t[\delta] : T[\delta] \circledR [\![t]\!]_\rho \in_\omega [\![T]\!]_\rho$. In order to show this, first we observe that $\Delta^{\text{🔓}} \vdash_m \delta : \Gamma \circledR \rho$ from Lemmas 4.4.3 and 4.4.5. Therefore, $\Delta.\text{🔒} \vdash_{m'} \delta : \Gamma.\text{🔒} \circledR \rho$ by definition. Finally, instantiating our induction hypothesis with this gives us our goal.

Case.

$$\frac{\Gamma \vdash A \textit{ type} \qquad \Gamma^{\text{🔓}} \vdash t : \Box T}{\Gamma \vdash [t]_{\text{🔓}} : T}$$

We have by induction hypothesis in this case that $\Gamma \vDash_n T$ type and $\Gamma.\text{🔒} \vDash_n t : T$. We wish to show $\Gamma \vDash_n [t]_{\text{🔒}} : \Box T$. For this, suppose we have $m \leq n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We wish to show the following:

$$\Delta \vdash_m [t]_{\text{🔓}}[\delta] : (\Box T)[\delta] \circledR [\![[t]_{\text{🔓}}]\!]_\rho \in_\omega [\![\Box T]\!]_\rho$$

We observe by Lemma 4.4.6 that $\Delta^{\text{🔓}} \vdash_m \delta : \Gamma^{\text{🔓}} \circledR \rho$. We therefore may instantiate our induction hypothesis to conclude the following:

$$\Delta^{\text{🔓}}.\text{🔒} \vdash_m [t[\delta]]_{\text{🔓}} : T' \circledR \underline{\mathbf{open}}(\llbracket t \rrbracket_\rho) \in_\omega \llbracket T \rrbracket_\rho$$

Where $\Delta^{\text{🔓}} \vdash \Box(T[\delta]) = \Box T'$ *type*. Now, by Lemmas 4.3.2 and 4.4.2 we have that this gives us the following:

$$\Delta \vdash_m [t[\delta]]_{\text{🔓}} : T' \circledR \underline{\mathbf{open}}(\llbracket t \rrbracket_\rho) \in_\omega \llbracket T \rrbracket_\rho$$

Now, from Corollary 4.3.12, our induction hypothesis, and calculation this gives us the goal:

$$\Delta \vdash_m [t]_{\text{🔓}}[\delta] : T[\delta] \circledR \llbracket [t]_{\text{🔓}} \rrbracket_\rho \in_\omega \llbracket T \rrbracket_\rho$$

Case.

$$\frac{\Gamma.\text{🔒} \vdash A : \mathsf{U}_i}{\Gamma \vdash \Box A : \mathsf{U}_i}$$

Identical to the case for $\Gamma \vdash \Box A$ *type*.

Case.

$$\frac{\Gamma \ ctx}{\Gamma \vdash \mathsf{U}_i : \mathsf{U}_{i+1}}$$

Identical to the case for $\Gamma \vdash \mathsf{U}_i$ *type*.

Case.

$$\frac{\Gamma \vdash A : \mathsf{U}_i}{\Gamma \vdash A : \mathsf{U}_{i+1}}$$

Identical to the case for $\Gamma \vdash \mathsf{U}_i$ *type*.

Case.

$$\frac{\Gamma \vdash \delta : \Delta \qquad \Delta \vdash t : A}{\Gamma \vdash t[\delta] : A[\delta]}$$

This case mirrors the case for $\Gamma \vdash T[\delta]$ *type*.

Case.

$$\frac{\Gamma \vdash A = B \ type \qquad \Gamma \vdash t : A}{\Gamma \vdash t : B}$$

Immediate from Lemma 4.3.6.

3. If $\Gamma \vdash \delta : \Delta$ then $\Gamma \vDash_n \delta : \Delta$ for any n.

Case.

$$\frac{\Gamma \ ctx}{\Gamma \vdash \cdot : \cdot}$$

For this, suppose we have $m \leq n$ and $\Delta \vdash_m \delta : \Gamma \circledR \rho$. We wish to show $\Delta \vdash_m \cdot \circ \delta : \Gamma \circledR \llbracket \cdot \rrbracket_\rho$. By calculation $\llbracket \cdot \rrbracket_\rho = \cdot$. The goal then follows by applying a rule.

Case.

$$\frac{\Gamma_1 \ ctx \qquad \Gamma_2 \ ctx \qquad \Gamma_1 \rhd_{\text{🔒}} \Gamma_2}{\Gamma_1 \vdash \mathsf{id} : \Gamma_2}$$

For this, suppose we have $m \leq n$ and $\Delta \vdash_m \delta : \Gamma_1 \circledR \rho$. We wish to show $\Delta \vdash_m \mathsf{id} \circ \delta : \Gamma_2 \circledR \llbracket \mathsf{id} \rrbracket_\rho$. By calculation, this is equivalent to $\Delta \vdash_m \mathsf{id}delta : \Gamma_2 \circledR \rho$. This is a result of Lemma 4.3.2.

Case.

$$\frac{\Delta \vdash T\ \mathit{type} \qquad \Gamma \vdash \delta : \Delta \qquad \Gamma \vdash t : T[\delta]}{\Gamma \vdash \delta.t : \Delta.T}$$

In this case, we have by induction hypothesis that $\Delta \vDash_n T$ type, $\Gamma \vDash_n \delta : \Delta$, and $\Gamma \vDash_n t : T[\delta]$. We wish to show $\Gamma \vDash_n \delta.t : \Delta.T$.

For this, suppose we have $m \leq n$ and $\Delta' \vdash_m \delta' : \Gamma \circledR \rho$. We wish to show the following:

$$\Delta' \vdash_m (\delta.t) \circ \delta' : \Delta.T \circledR [\![\delta]\!]_\rho.[\![t]\!]_\rho$$

By calculation, it suffices to show the following:

$$\Delta' \vdash_m (\delta \circ \delta').t[\delta'] : \Delta.T \circledR [\![\delta]\!]_\rho.[\![t]\!]_\rho$$

In order to do this, we merely need to show $\Delta' \vdash_m \delta \circ \delta' : \Delta \circledR [\![\delta]\!]_\rho$, $\tau_\omega \models_n [\![T]\!]_\rho \sim [\![T]\!]_\rho$, and $\Delta' \vdash_m t[\delta'] : T[\delta \circ \delta'] \circledR [\![t]\!]_\rho \in_\omega [\![T[\delta]]\!]_\rho$. The second is a result of Theorem 3.3.5 and the remaining two are immediate from our induction hypothesis.

Case.

$$\frac{\Gamma_1 \vdash \delta_1 : \Gamma_2 \qquad \Gamma_2 \vdash \delta_2 : \Gamma_3}{\Gamma_1 \vdash \delta_2 \circ \delta_1 : \Gamma_3}$$

In this case, we have by induction hypothesis that $\Gamma_1 \vDash_n \delta_1 : \Gamma_2$, and $\Gamma_2 \vDash_n \delta_2 : \Gamma_3$. We wish to show $\Gamma_1 \vDash_n \delta_2 \circ \delta_1 : \Gamma_3$.

We assume we have $m \leq n$ and $\Gamma_0 \vdash_m \delta' : \Gamma_1 \circledR \rho$. We then have $\Gamma_0 \vdash_m \delta_1 \circ \delta' : \Gamma_2 \circledR [\![\delta_1]\!]_\rho$. We then have the following:

$$\Gamma_0 \vdash_m (\delta_2 \circ \delta_1) \circ \delta' : \Gamma_3 \circledR [\![\delta_2]\!]_{[\![\delta_1]\!]_\rho}$$

Calculation tells us that $[\![\delta_2]\!]_{[\![\delta_1]\!]_\rho} = [\![\delta_2 \circ \delta_1]\!]_\rho$ finishing this case.

Case.

$$\frac{\Gamma_1\ \mathit{ctx} \qquad \Gamma_1^{\text{🔓}} \vdash \delta : \Gamma_2}{\Gamma_1 \vdash \delta : \Gamma_2.\text{🔒}}$$

In this case, we have by induction hypothesis that $\Gamma_1^{\text{🔓}} \vDash_n \delta : \Gamma_2$ and we wish to show $\Gamma_1 \vDash_n \delta : \Gamma_2.\text{🔒}$.

We assume we have $m \leq n$ and $\Gamma_0 \vdash_m \delta' : \Gamma_1 \circledR \rho$. We then have that there is some m' such that $\Gamma_0^{\text{🔓}} \vdash_{m'} \delta' : \Gamma_1^{\text{🔓}} \circledR \rho$ by Lemma 4.4.6. We then have $\Gamma_0^{\text{🔓}} \vdash_{m'} \delta \circ \delta' : \Gamma_2 \circledR [\![\delta]\!]_\rho$. Therefore, by definition we have $\Gamma_0 \vdash_{m'} \delta \circ \delta' : \Gamma_2.\text{🔒} \circledR [\![\delta]\!]_\rho$ as required.

Case.

$$\frac{\Gamma_1.\Gamma_2\ \mathit{ctx} \qquad \Gamma_1'\ \mathit{ctx} \qquad \Gamma_1 \rhd_{\text{🔒}} \Gamma_1' \qquad k = \|\Gamma_2\| \qquad \text{🔒} \notin \Gamma_2}{\Gamma_1.\Gamma_2 \vdash \mathsf{p}^k : \Gamma_1'}$$

Suppose we have $m \leq n$ and $\Delta \vdash_m \delta : \Gamma_1.\Gamma_2 \circledR \rho$. We wish to show $\Delta \vdash_m \mathsf{p}^k \circ \delta : \Gamma_1' \circledR \rho$. This follows by Lemma 4.4.3. □

Lemma 4.4.8. *If Γ ctx and $\uparrow\Gamma = \rho$ then $\Gamma \vdash_n \mathsf{id} : \Gamma \circledR \rho$.*

Proof. We proceed by induction on Γ *ctx*.

Case.

$$\frac{}{\cdot\ \mathit{ctx}}$$

In this case we must show that $\cdot \vdash_n \mathsf{id} : \cdot \circledR \cdot$. This is immediate as $\cdot \vdash \mathsf{id} : \cdot$.

Case.

$$\frac{\Gamma\ ctx}{\Gamma.🔒\ ctx}$$

In this case we have by induction hypothesis that $\Gamma \vdash_n \mathsf{id} : \Gamma \circledR \rho$ where $\uparrow\Gamma = \rho$. We therefore must show that $\Gamma.🔒 \vdash_n \mathsf{id} : \Gamma.🔒 \circledR \rho$. We have by Lemma 4.4.3 that $\Gamma^{🔓} \vdash_n \mathsf{id} : \Gamma \circledR \rho$ holds and so we have the desired conclusion by definition.

Case.

$$\frac{\Gamma\ ctx \qquad \Gamma \vdash T\ type}{\Gamma.T\ ctx}$$

In this case we have by induction hypothesis that $\Gamma \vdash_n \mathsf{id} : \Gamma \circledR \rho$ where $\uparrow\Gamma = \rho$. We therefore must show that $\Gamma.T \vdash_n \mathsf{id} : \Gamma.T \circledR \rho.\mathsf{var}_{\|\Gamma\|}$. First, we observe that it suffices to show $\Gamma.T \vdash_n \mathsf{p}^1.\mathsf{var}_0 : \Gamma.T \circledR \rho.\mathsf{var}_{\|\Gamma\|}$. Now, from $\Gamma \vdash T\ type$ we may conclude that $\Gamma.T \vDash_n \mathsf{var}_0 : T[\mathsf{p}^1]$. Therefore, we have some A such that $\tau_\omega \models_n A \sim A \downarrow R$, $[\![T]\!]_\rho = A$, and $\Gamma.T \vdash_n \mathsf{var}_0 : T[\mathsf{p}^1] \circledR \uparrow^A \mathbf{var}_{\|\Gamma\|} \in_\omega A$.

Next, we observe that by Lemma 4.4.3 that $\Gamma.T \vdash_n \mathsf{p}^1 : \Gamma \circledR \rho$ holds and so we have the desired conclusion by definition.

□

Corollary 4.4.9. *If* $\Gamma \vdash t : T$ *and* $\underline{\mathbf{nbe}}_\Gamma^T(t) = t'$ *then* $\Gamma \vdash t = t' : T$.

Proof. From Theorem 4.4.7 we have that $\Gamma \vDash_n t : T$. Therefore, by Lemma 4.4.8 we have that $\Gamma \vdash_n t : T \circledR [\![t]\!]_\rho \in_\omega [\![T]\!]_\rho$ where $\uparrow\Gamma = \rho$. From Lemma 4.3.11, then, we have that $\lceil \downarrow^{[\![T]\!]_\rho} [\![t]\!]_\rho \rceil_{\|\Gamma\|} = t'$ such that $\Gamma \vdash t = t' : T$. This gives the desired goal. □

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