Universes in Simplicial Type Theory

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Aarhus University

Simplicial type theory

Starting point: homotopy type theory

Since ~2010, type theorists have spent a lot of time thinking about HoTT:

- Gives us a dictionary (semantic model) between MLTT and homotopy theory

Type Theory	Homotopy Theory
type	space
dependent type	fibration
term	point
$p : \mathrm{Id}(A, a, b)$	path in a space

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- We can import principles from this model into MLTT: univalence.

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Homotopy type theory II

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Univalence...

- ensures that every operation is invariant under equivalence.
- gives us access to the structure-identity principle.
- can let us mirror some parametricity-type arguments.
- improves the behavior of quotients, etc.

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In summary, HoTT is neat and we'll now consider some more of it.

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While HoTT is nice, there is one serious cost:

We can no longer just say “X is equal to Y” and be done with it...

...we must also explain *how* they’re equal.

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Quite a pain when doing non-truncated algebra in HoTT

Definition

A (coherent/∞-)*monoid* is a type A together $\epsilon : A$ and $\cdot : A \times A \to A$ where

- idl : $\epsilon \cdot a = a$, idr : $b \cdot \epsilon = b$.
- assoc : $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- And an infinite tower of additional equations *between* idl *and friends*.

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Meet: the coherence problem

A coherence Urproblem: ∞-categories

Idea Solve ∞-categories/functors and bootstrap everything from that.

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For now the ∞ can be silent

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And thus we arrive at… directed homotopy type theory!

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Could get here lots of other ways... many motivations for wanting a type theory for category theory!

A new problem (hurray)

Well, we've replaced our coherence problem with a new problem:

How do we design a type theory where types are categories?

Directed Homotopy Type Theory	∞-Category Theory
type A	∞-category $\mathcal{C}$
term $a : A$	object $c : \mathcal{C}$
function $f : A \to A'$	$F : \mathcal{C} \to \mathcal{C}'$.
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a type $\mathbb{I}$ and maps $\mathbb{I} \to A$	$\{0 \leqslant 1\}$ and $\mathcal{C}^{\to}$

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Core simplicial type theory

Core simplicial type theory = HoTT + a postulated $\mathbb{I}$

STT Axiom A

$\mathbb{I}$: HSet with $0, 1 : \mathbb{I}$ and $\leqslant\, : \mathbb{I} \times \mathbb{I} \to$ HProp making $\mathbb{I}$ a bounded total order.

We imagine $\mathbb{I} \sim \{0 \to 1\}$ as a category, so we can now define[1]

$$\mathrm{Hom}_A(x, y) = \textstyle\sum_{f : \mathbb{I} \to A} f(0) = x \times f(1) = y \qquad \mathrm{id}_x = (\lambda\_.\, x, \mathrm{refl}, \mathrm{refl}) : \textstyle\prod_x \mathrm{Hom}_A(x, x)$$

[1] Be warned! Revisionism!

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Ouch, most types (1) do not admit composition and (2) do not identify Id with $\simeq$.

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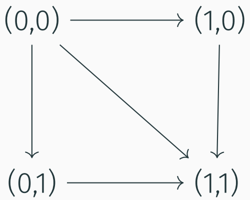
Isolating categories among types: the basic shapes

First: find those types which do believe they're categories.

$$\Delta^2 = \{(i,j) : \mathbb{I} \times \mathbb{I} \mid i \geqslant j\} \qquad \Lambda^2_1 = \{(i,j) : \mathbb{I} \times \mathbb{I} \mid i = 1 \vee j = 0\}$$

The walking square $\mathbb{I} \times \mathbb{I}$



A map $\tau : \Delta^2 \to A$ *witnesses* the composition $\tau(1,-) \circ \tau(-,0)$ is $\lambda i.\, \tau(i,i)$.

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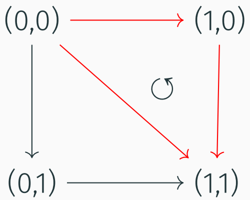
The walking composed triangle



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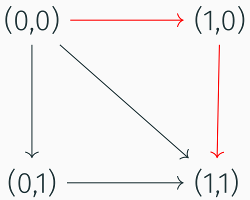
The walking composition problem



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The basic definitions

A type is *Segal* (has coherent compositions) if...

$$\text{isEquiv}(X^{\Delta^2} \to X^{\Lambda^2_1})$$

A Segal type is Rezk (is univalent) if isomorphism and equality agree:

$$\text{isEquiv}(X \to \text{Isos}(X))$$

Definition

A **category** is a Segal and Rezk type. A **groupoid** is a category with only isos.

Remarkable facts

(1) This is a mere property! (2) It encodes all of the infinite coherences!

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isomorphism = path; ∞-category theory!

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While not every type is a category...

Lemma (functions are functors)

If $f : A \to B$ and A and B are both categories, f is a functor.

Proof

Functorial action is post-composition $\lambda x. f \circ x : (\mathbb{I} \to A) \to (\mathbb{I} \to B)$.

Preservation of identities

$$f \circ \mathrm{id}_x = f \circ \mathrm{const}\, x = \mathrm{const}(f\, x) = \mathrm{id}_{f\, x}$$

Preservation of composition: if $\tau : \Delta^2 \to A$ then

$$f \circ \tau \stackrel{?}{=} \mathrm{compose}(f \circ \tau|_{\Lambda^2_1}) : \text{“Solutions to comp. problem } f \circ \tau|_{\Lambda^2_1}\text{”}$$

... but solutions are unique.

Everything being functorial is sometimes too restrictive...

Simplicial Type Theory	∞-Category Theory
$\langle\flat \mid A\rangle$	the groupoid core $\mathcal{C}^{\simeq}$
$\langle\mathrm{op} \mid A\rangle$	the opposite $\mathcal{C}^{\mathrm{op}}$

[2]... and then bug me to write something better.

Modalities in simplicial type theory

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$\langle \flat \mid A \rangle$	the groupoid core $\mathcal{C}^{\simeq}$
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Don't want to spend a lot of time on modalities. Read my thesis[2]

Rule of thumb

(1) annotate some variables $x :_\flat A$

(2) if $a : A$ only uses $\flat$-variables, $\mathrm{mod}_\flat(a) : \langle \flat \mid A \rangle$.

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Axioms in simplicial type theory

Lots of axioms; we focus on some straightforward consequences:

Lemma

$\mathbb{I}$ is a category and $\langle \flat \mid \mathbb{I} \rangle = \mathrm{Bool}$.

Lemma (fundamental theorem of category theory)

Given $\flat$-categories A, B and $f :_\flat A \to B$, f is an equivalence iff

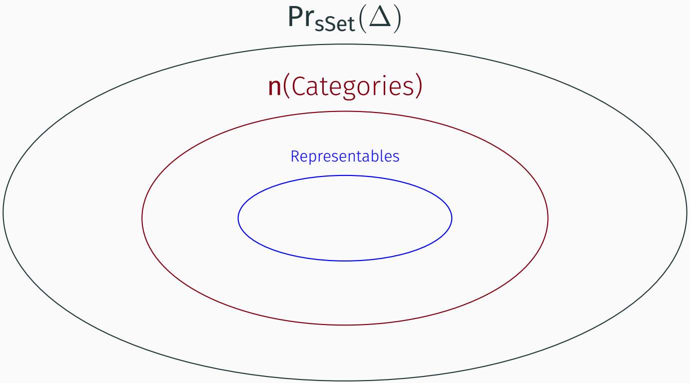
1. If $y :_\flat B$ there exists $x : A$ such that $f(x) \cong y$
2. If $x, x' :_\flat A$ then $\mathrm{isEquiv}(f_* : \mathrm{Hom}(x, x') \to \mathrm{Hom}(f(x), f(x')))$.

Lemma

If $\flat$-categories A, B and $\alpha :_\flat \mathbb{I} \to (A \to B)$ then $\mathrm{isIso}(\alpha)$ iff $\prod_{x :_\flat A} \mathrm{isIso}(\alpha(-, x))$.

What is this all supposed to mean?

Simplicial type theory is justified by the semantic model in $\mathbf{Pr}_{\mathsf{sSet}}(\Delta)$ [Shu15]:



Concisely, $\mathbf{n} : \mathbf{Cat} \to \mathbf{Pr}(\Delta)$ embeds (∞-)categories fully-faithfully into $\mathbf{Pr}(\Delta)$.

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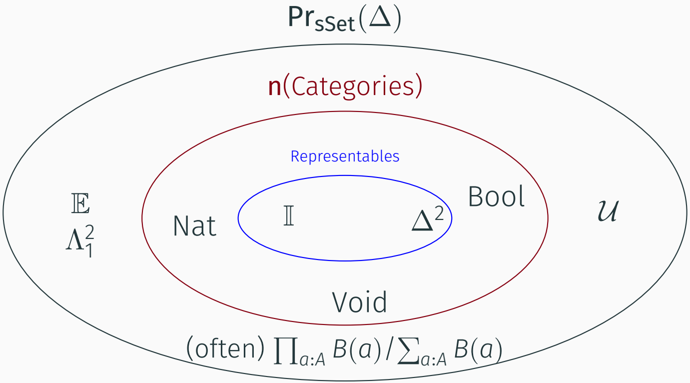

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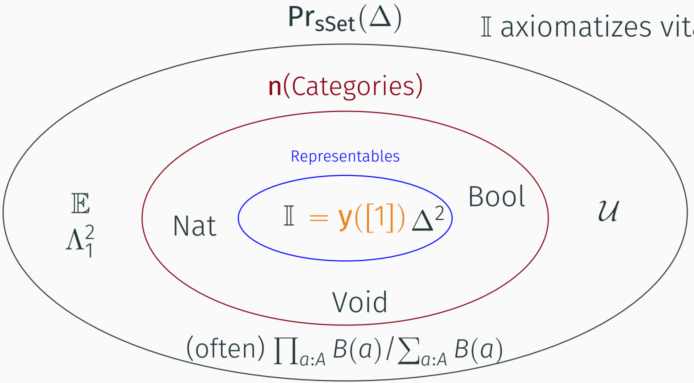

$\mathbb{I}$ axiomatizes vital representable $\mathbf{y}(\{0 \leqslant 1\})$

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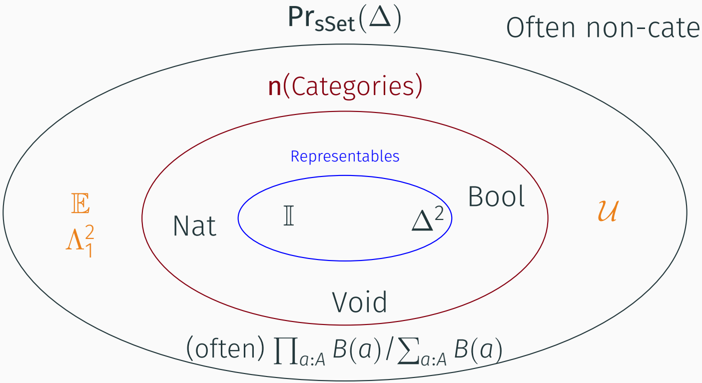

Often non-categories are vital anyways!

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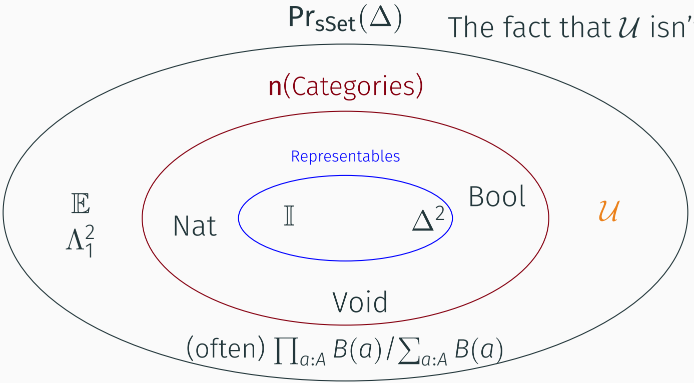

The fact that $\mathcal{U}$ isn't a category leads to this talk.

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What remains?

Of course, quite a lot is left to be done. Two big TODOs:

1. Define/characterize the ∞-categories of spaces and categories
2. Non-trivial generating ∞-categories to build things up

We'll focus on (1). It turns out to be very useful for (2).

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???	the category of spaces
???	the category of categories

Strange universes in STT

Our task

We want to fill out the ???s.

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How do we know when we have succeeded? Consider $\mathcal{S}$:

- Clearly want $\mathcal{S}$ to be... a category.
- Should have $\langle \flat \mid \mathcal{S} \rangle \simeq$ groupoid of small groupoids
- What about the rest of the structure?

A universal property for $\mathcal{S}$: straightening–unstraightening

To be sure that $\mathcal{S}$ is correct, we characterize it by a universal property:

$$\mathsf{unStraighten} : \langle \flat \mid X \to \mathcal{S} \rangle \simeq \langle \flat \mid \mathsf{CovFam}(X) \rangle$$

(Compare with ordinary universe where $(X \to \mathcal{U}) \simeq \mathsf{Fam}(X)$)

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$$\langle \flat \mid \mathsf{CovFam}(\mathbf{1}) \rangle \simeq \langle \flat \mid \mathcal{U}_{\mathsf{isGrpd}} \rangle \qquad \langle \flat \mid \mathsf{CovFam}(\mathbb{I}) \rangle \simeq \langle \flat \mid \textstyle\sum_{A,B:\mathcal{U}_{\mathsf{isGrpd}}} A \to B \rangle$$

- By Yoneda game, $\mathcal{S}$ is a subtype of $\mathcal{U}$.
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Critical difference: the maps $\mathbb{I} \to \mathcal{S}$ are very different than maps $\mathbb{I} \to \mathcal{U}_{\mathsf{isGrpd}}$.

Future Theorem (Directed Univalence)

A map $\mathbb{I} \to \mathcal{S}$ is equivalent to $\sum_{A,B:\mathcal{S}} A \to B$.

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Before we prove all this, let's see what it lets us do...

A fun little aside

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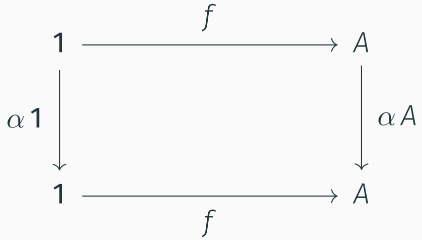
Fix $\alpha : \prod_{A:\mathcal{S}} \mathrm{Hom}_{\mathcal{S}}(A, A)$ and $x : A$

- $f = \lambda \star . x$ induces morphism $\mathrm{Hom}_{\mathcal{S}}(\mathbf{1}, A)$.
- $\alpha \circ f : \mathbb{I} \to (\mathbb{I} \to \mathcal{S})$ induces commuting square



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- $f = \lambda \star .x$ induces morphism $\mathrm{Hom}_{\mathcal{S}}(\mathbf{1}, A)$.
- $\alpha \circ f : \mathbb{I} \to (\mathbb{I} \to \mathcal{S})$ induces commuting square

$$x = (f \circ \alpha\,\mathbf{1})(\star)$$

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{\;f\;} & A \\ {\scriptstyle \alpha\,\mathbf{1}}\downarrow & & \downarrow{\scriptstyle \alpha\,A} \\ \mathbf{1} & \xrightarrow[\;f\;]{} & A \end{array}$$

A fun little aside

Theorem

The type of functions $\prod_{A:\mathcal{S}}(A \to A)$ is contractible.

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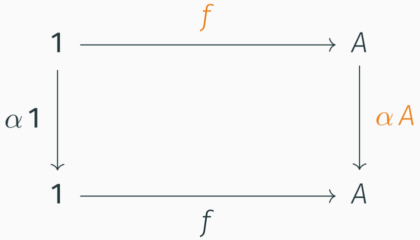

Defining covariant families

Time to pay the piper and start defining things.

$$\text{isCov}(A : X \to \mathcal{U}) = \prod_{x:\mathbb{I}\to X} \prod_{a_0:A(x\,0)} \text{isContr}\left(\sum_{a:(i:\mathbb{I})\to A(x\,i)} a\,0 = a_0\right)$$

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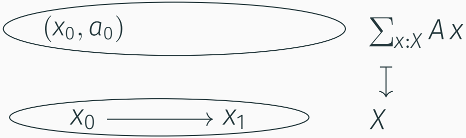

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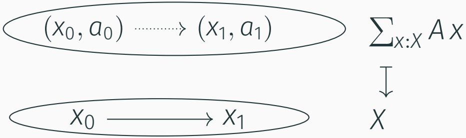

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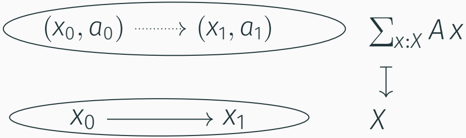

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“just” Alert!

Just orthogonality $\{0\} \to \mathbb{I} \quad \pitchfork \quad (\sum_{x:X} A\,x) \to X$

A possible approach: LOPS

New (more precise) goal

$\mathcal{S}$ should be the base of the universal covariant family.

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New hope

(Homotopy) type theorists have some experience building universal fibrations!

The Licata–Orton–Pitts-Spitters [Lic+18] recipe

- Define the fibrancy structure as a map $\mathcal{U}^{\mathbb{I}} \to \mathcal{U}$
- use amazing right adjoint to $(-)^{\mathbb{I}}$ to transpose this and define $\mathcal{V}$
- Draw the rest of the owl: prove $\mathcal{V}$ fibrant & closed under connectives

Successfully applied to *bicubical* type theory by Weaver and Licata [WL20].

Can we just do LOPS for simplicial type theory?

Unfortunately, this doesn't work for our setting.

- Our intended model is $\mathsf{Pr}(\Delta)$ where $(-)^{\mathbb{I}}$ is simply not a left adjoint.
- Why not? $\mathbb{I}$ is still representable, but Δ is not closed under products.

Have to work a bit harder.

We're gonna need a bigger ~~boat~~ category

For this, we turn to $\mathbf{Pr}(\square)$, (Dedekind) cubical spaces.

Sattler, Kapulkin & Voevodsky, Streicher & Weinberger

There is an essential geometric embedding $\mathbf{Pr}(\Delta) \longrightarrow \mathbf{Pr}(\square)$.

What it means for us:

TTT Axiom A

$\mathbb{I}$: HSet with bounded distributive lattice structure $0, 1 : \mathbb{I}$ and $\wedge, \vee : \mathbb{I}^2 \to \mathbb{I}$.

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TTT Axiom E

There is an amazing right adjoint $(-)^{\mathbb{I}} \dashv (-)_{\mathbb{I}}$ for $\flat$-types.

Relating triangulated type theory and simplicial type theory

Q. We're morally in $\mathbf{Pr}(\square)$, how do we isolate types coming form $\mathbf{Pr}(\Delta)$?

A. It's a subtopos, so an RSS modality [RSS20]!

Slogan: simplicial types believe the interval is totally ordered

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Slogan: simplicial types believe the interval is totally ordered

Let us write $⧅$ for the lex modality classifying types with the following property:

$$\prod_{i,j:\mathbb{I}} \mathsf{isEquiv}\left(A \longrightarrow A^{i \leqslant j \vee j \leqslant i}\right)$$

Example

$\mathbb{I}$ is simplicial.

Open question from Weaver–Licata (maybe solved now by Rob and me!?)

If A is Segal (and Rezk, why not), is it simplicial?

Back to our regularly scheduled modal suffering

Back to our task, we begin with the right notion of fibrancy structure.

$$\mathsf{covStruct}(A : \mathcal{U}^{\mathbb{I}}) = \prod_{a_0 : A\,0} \mathsf{isContr}(\sum_{a : (i:\mathbb{I}) \to A\,i} a\,0 = a_0)$$

This the generic case (a family over $\mathbb{I}$) of what we saw earlier

Why bother? Well, we have some moves available for maps $\mathcal{U}^{\mathbb{I}} \to \mathcal{U}$...

The universe of spaces pt 1

For general nonsense reasons, we have a chain of maps:

$$\begin{aligned}\langle \flat \mid \mathcal{U}^{\mathbb{I}} \to \mathcal{U} \rangle &\simeq \langle \flat \mid \mathcal{U} \to \mathcal{U}_{\mathbb{I}} \rangle \\ &\to \langle \flat \mid \mathcal{U} \to \mathcal{U} \rangle\end{aligned}$$

Feeding this covStruct, we obtain aCovStruct : $\mathcal{U} \to \mathrm{HProp}$

Daniel's intuition 🤷

aCovStruct(A) roughly means "A is covariant in the whole context".

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$$\to \langle \flat \mid \mathcal{U} \to \mathcal{U} \rangle \qquad \text{have a map } \mathcal{U}_{\mathbb{I}} \to \mathcal{U} \text{ classifying } (\mathcal{U}_\bullet)_{\mathbb{I}} \to (\mathcal{U})_{\mathbb{I}}$$

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The only theorem you ever need about aCovStruct

If $A :_\flat X \to \mathcal{U}$:

$$\langle \flat \mid (x : X) \to \mathrm{aCovStruct}(A\,x) \rangle \iff \langle \flat \mid \mathrm{isCov}(A : X \to \mathcal{U}) \rangle$$

The universe of spaces pt 2

We can now define the putative universe of spaces!

$$\mathcal{S} = \sum_{A:\mathcal{U}_{\boxslash}} \mathsf{aCovStruct}(A)$$

Theorem

The map $\mathcal{S} \to \mathcal{U}$ induces $\langle \flat \mid X \to \mathcal{S} \rangle \simeq \langle \flat \mid \sum_{A:X\to\mathcal{U}_{\boxslash}} \mathsf{isCov}(A) \rangle = \langle \flat \mid \mathsf{CovFam}(A) \rangle$

Proof

Both $\flat$ and $-^X$ commute with $\sum$-types. Unfold and apply Theorem Of aCovStruct.

The rest of the owl

As mentioned earlier, next is to actually show that $\mathcal{S}$ is a well-behaved universe:

- We want closure under various connectives and operations
- We want to characterize the type of arrows in $\mathcal{S}$
- We want to know that $\mathcal{S}$ is an ∞-category

Building amazingly covariant families

It turns out that closing $\mathcal{S}$ under connectives is OK!

Lemma

$\mathcal{S}$ is closed under $\mathbf{1}$, Σ, Bool, and $=$

Harder Lemma

If $A : \langle \mathrm{op} \mid \mathcal{S} \rangle$ and $B : \mathcal{S}$ then $\langle \mathrm{op} \mid A \rangle \to B : \mathcal{S}$

Easier Lemma

The predicate $- = 1$ induces a map $\mathbb{I} \to \mathcal{S}$

Directed univalence

Theorem

The following map sending $\mathcal{S}^{\mathbb{I}}$ to $\sum_{A,B:\mathcal{S}} A \to B$ is an equivalence:

$$U(A) = (A(0), A(1), \mathrm{coe}_A)$$

💭: construct a quasi-inverse by sending (A, B, f) to the “directed glue type”:

$$\mathrm{Gl}(A, B, f)\ i = \sum_{b:B} i = 0 \to f^{-1}(b)$$

This lands in $\mathcal{S}$ by closure properties!

$U(\mathrm{Gl}(p)) = p$ by computation, $\mathrm{Gl}(U(A)) = A$ by lemma.

Segalness

Using similar arguments, one can show the following

Theorem

$\mathcal{S}$ is Segal

Proof

Argue $\mathcal{S}^{\Delta^2} = \sum_{A,B,C:\mathcal{S}}(A \to B) \times (B \to C)$ by generalizing prior proof.

Corollary

$\mathcal{S}$ is Rezk

A complication: is $\mathcal{S}$ simplicial?

We want to show that $\mathcal{S}$ is a category (i.e., is simplicial).

Uh oh Remember that we built $\mathcal{S}$ using $(-)_{\mathbb{I}}$ which is not always simplicial!

Hmm Does it happen that $\mathcal{S}$ is simplicial anyways?

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Consider the generic morphism: $\pi : \mathcal{S}^{\bullet} = \sum_{A:\mathcal{S}} A \longrightarrow \mathcal{S}$.

Lemma

If $⧅\pi$ is covariant, then $\mathcal{S}$ is simplicial.

Lemma

$⧅\pi$ is covariant.

$\mathcal{S}$: the final slide

Big Theorem

$\mathcal{S} \hookrightarrow \mathcal{U}$ is a category classifying covariant fibrations and satisfying directed UA.

Phew...

Strange(r) universes in STT

Hurray! We have $\mathcal{S}$. What's next?

No time to celebrate....

Simplicial Type Theory	∞-Category Theory
$\mathcal{S}$	the category of spaces
???	the category of categories

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Now on to Cat...

The plan for Cat

Optimism: Actually this is more-or-less a repetition of what we've seen with $\mathcal{S}$.

Realism: … with the wrinkle that everything is more poorly behaved for Cat.

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1. $\mathrm{Hom}_{\mathrm{Cat}}(A, B)$ is always a groupoid.
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The main problem: straightforward directed univalence can *never* be true for Cat:

1. $\mathrm{Hom}_{\mathrm{Cat}}(A, B)$ is always a groupoid.
2. $A \to B$ is basically never a groupoid.
3. Only get $\langle \flat \mid \mathrm{Hom}_{\mathrm{Cat}}(A, B)\rangle = \langle \flat \mid A \to B\rangle$ for $A, B :_\flat \mathrm{Cat}$.

Consequently: lots more has to be done $\flat$-annotated.

Our new universal property: coCartesian families

Just like with $\mathcal{S}$, Cat will have a universal property:

$$\langle \flat \mid X \to \mathsf{Cat} \rangle \simeq \langle \flat \mid \mathsf{coCartFam}(X) \rangle$$

- Same Yoneda game: $\mathsf{Cat} \hookrightarrow \mathcal{U}$.
- Cocartesian families $\sim$ coherently functorial families of categories.
- Developed extensively in STT by Buchholtz and Weinberger [BW23].

Slogan

Covariant fibrations are to groupoids as cocartesian fibrations are to categories.

Two definitions of coCartesian families

Fix a family of categories $A : X \to \mathcal{U}$.

Recall, $A : X \to \mathcal{U}$ is covariant iff $\prod_{x:X^{\mathbb{I}}} \prod_{a_0:A(x\,0)} \mathrm{isContr}(\sum_{a:(i:\mathbb{I})\to X\,i} a\,0 = a_0)$

Initial sections definition

A is coCartesian if $\sum_{a:(i:\mathbb{I})\to A(x\,i)} a\,0 = a$ has an initial object $(x : X^{\mathbb{I}}, a : A(x\,0))$[3]

Just as before: gives us coherent transport maps $A(x_0) \to A(x_1)$.

[3] Thanks to Lossin (arXiv:2604.18668) & Schellingerhout! See forthcoming masters theses.

LOPS for coCartesian families

Just as we used $(-)_{\mathbb{I}}$ to define aCovStruct, we can play the same game:

Definition/Theorem

There is a (necessarily unique) $\mathsf{aCoCartStruct} : \mathcal{U} \to \mathsf{HProp}$ such that if $A :_\flat X \to \mathcal{U}$

$$\langle \flat \mid (x : X) \to \mathsf{aCoCartStruct}(A(x)) \rangle \simeq \langle \flat \mid A \text{ is cocartesian} \rangle$$

Now we can complete the story:

$$\mathsf{Cat} = \textstyle\sum_{A:\mathcal{U}_{\boxslash}} \mathsf{aCoCartStruct}(A)$$

Theorem

For any $X :_\flat \mathcal{U}$, we have $\langle \flat \mid X \to \mathsf{Cat} \rangle \simeq \langle \flat \mid \mathsf{coCartFam}(X) \rangle$.

Directed univalence and its ilk

Again, with some modal chicanery the story goes through unchanged from $\mathcal{S}$

Theorem (directed univalence for Cat)

There is a canonical equivalence $\langle \flat \mid \mathbb{I} \to \mathsf{Cat} \rangle \simeq \langle \flat \mid \sum_{A,B:\mathcal{U}_{\mathsf{isCat}}} A \to B \rangle$.

Theorem

Cat is Segal and Rezk.

Theorem

Cat is simplicial.

New trick: transform problem into (more complex) $\flat$-problem.

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And now... breath. 🧘

Vistas

Lots of other stuff in the pipeline

Lots done...

- Riehl and Shulman [RS17]: foundations, covariant fibrations, adjunctions
- Weinberger and Buchholtz [Wei22; BW23; Wei24]: fibered ∞-category theory
- Bardomiano Martínez [Bar23]: limits, colimits, exponentiable fibrations.

Lots being done

- The Yoneda embedding, Kan extensions, cocompletions, cofinality, etc.
- Stable ∞-categories, operads/higher algebra (as seen this week by Rob)
- Filtered and sifted colimits, decomposition of (co)limits
- Exponentiable maps in STT/Cisinski et al.'s Axiom M
- More fibered category theory (Lossin)
- Hopelessly incomplete...

Also more CS applications (jww Chhabra & Angiuli).

Formalization: a challenge to myself

Impressive amounts of STT (all of [RS17] and much more) formalized in Rzk

`rzk-lang.zulipchat.com`

Also lots of interesting (modal) stuff happening in Agda

github.com/samtoth/agda-synthetic-categories

A personal goal: let's formalize all of these arguments.

Come join us

Lots of work to do! Some of it is even fun!

- Something to read: *A type theory for synthetic ∞-categories*[4]
- Somewhere to chat: `https://rzk-lang.zulipchat.com/`
- Somewhere to tinker: `https://rzk-lang.github.io/`
- Someone to collaborate with: I'm around all day

[4] I also like Discworld series by Terry Pratchett, but more globular than simplicial...

Conclusions

Simplicial Type Theory	∞-Category Theory
Segal and Rezk type A	∞-category $\mathcal{C}$
term $a : A$	object $c : \mathcal{C}$
function $f : A \to A'$	$F : \mathcal{C} \to \mathcal{C}'$.
$p : \mathrm{Id}(A, a, b)$	$\iota : c \cong c'$
a type $\mathbb{I}$ and maps $\mathbb{I} \to A$	$\{0 \leqslant 1\}$ and $\mathcal{C}^{\to}$
$\langle \flat \mid A \rangle$	the groupoid core $\mathcal{C}^{\simeq}$
$\langle \mathrm{op} \mid A \rangle$	$\mathcal{C}^{\mathrm{op}}$
$\mathcal{S}$	the category of spaces
Cat	the category of categories

https://arxiv.org/abs/2407.09146

https://arxiv.org/abs/2602.02218

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