Category Theory
Fall 2020

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Assignment 6
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Exercise 1. Let $\mathbb{C}$ and $\mathbb{D}$ be two categories and assume $\mathbb{D}$ has pullbacks. Let $F, G : \mathbb{C} \to \mathbb{D}$ be functors. Show that a natural transformation $\alpha : F \to G$ is a monomorphism *if and only if* each of the components $\alpha_c : F(c) \to G(c)$ is a monomorphism in $\mathbb{D}$.

Exercise 2. Let **FinSets** be the category of finite sets and functions. Let ω be the ordered set of natural numbers with the usual order

$$0 \leq 1 \leq 2 \leq 3 \leq \cdots$$

1. Show that the category $\mathbf{FinSets}^{\omega^{\mathrm{op}}}$ *is* cartesian closed.

   Hint: follow the general construction of exponents in $\mathbf{Sets}^{\omega^{\mathrm{op}}}$ and show it restricts to $\mathbf{FinSets}^{\omega^{\mathrm{op}}}$.

2. Show that the category $\mathbf{FinSets}^{\omega}$ *is not* cartesian closed.
   Hint: Consider the object N defined as

   $$[0] \hookrightarrow [1] \hookrightarrow [2] \hookrightarrow [3] \hookrightarrow \cdots$$

   where $[n]$ is the set $\{0, 1, \ldots n\}$ and all arrows are subset inclusions. Then consider the sets $\mathbf{Hom}(N, \mathbf{2})$ and $\mathbf{Hom}\left(\mathbf{1}, \mathbf{2}^N\right)$ assuming the exponential object $\mathbf{2}^N$ exists. The object $\mathbf{2}$ is as usual $\mathbf{1} + \mathbf{1}$.

You may assume that all *finite* limits exist in both $\mathbf{FinSets}^{\omega^{\mathrm{op}}}$ and $\mathbf{FinSets}^{\omega}$ and that they are given pointwise as in $\mathbf{Sets}^{\omega^{\mathrm{op}}}$ and $\mathbf{Sets}^{\omega}$.

Remark 1. The second item of the preceding exercise shows that even if $\mathbb{D}$ is cartesian closed and has all finite limits it need not be the case that $\mathbb{D}^{\mathbb{C}}$ is cartesian closed.

Exercise 3. Let **Sets** be the category of sets and functions and A a set. Does the functor

$$\mathbf{Hom}(A, -) : \mathbf{Sets} \to \mathbf{Sets}$$

have a left adjoint? Does it have a right adjoint? If they exist describe them, and if not prove it.
